



Asymptotic expansions for the relativistic Fermi-Dirac integral

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A few words of introduction

**Our field of expertise
is the computation of
Special Functions.**

**We are involved in the
DLMF project**

<http://dlmf.nist.gov>



Methods of computation for mathematical functions:

Elementary functions (trigonometric functions, exponential, log):

algorithms based on polynomial approximation and/or table lookup; Shift-and-Add algorithms.

Non-elementary functions:

rational approximations, asymptotic approximations, recurrence relations, numerical quadrature, ...



Examples of computation

We have considered the computation of:

1. Various types of Legendre functions of real parameters.
2. Homogeneous and inhomogeneous Airy functions.
3. Solution of the Bessel equation $x^2 y'' + xy + (x^2 + a^2)y = 0$ (2004). Numerically satisfactory pair $\{K_{ia}(x), L_{ia}(x)\}$.
4. Solution of the parabolic cylinder equation $y'' + (a \pm x^2/4)y = 0$ (2006 for $-$, 2011 and 2012 for $+$). Pairs of solutions $\{U(a, x), V(a, x)\}$ ($-$ case) and $\{W(a, x), W(a, -x)\}$
5. Conical functions, that is, Legendre functions $P_{-1/2+i\tau}^m(x)$.
6. Incomplete gamma functions $\gamma(a, x)$, $\Gamma(a, x)$.
7. Etc.

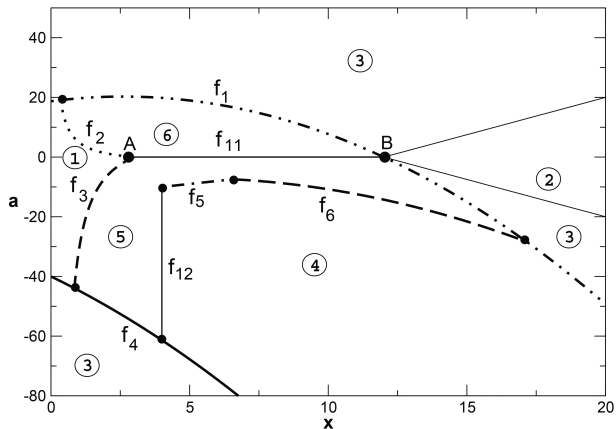


When the function depends on two or more variables a good number of methods are usually needed to cover a large range.



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Example: **case of the PCFs** $U(a, x), V(a, x)$



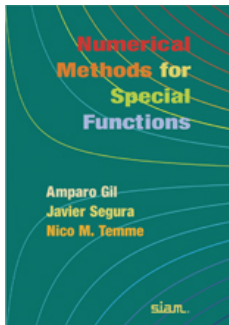


Methods used:

1. Maclaurin series.
2. Asymptotic expansions for large x (Poincaré type).
3. Uniform asymptotic expansions in terms of elementary functions.
4. Uniform Airy-type asymptotic expansion.
5. Integral representations.
6. Series for V and recurrences for U .

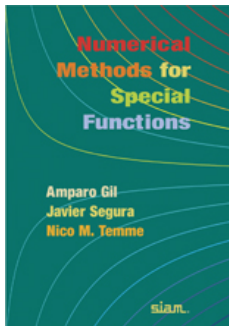


Details about methods to evaluate non-elementary functions are given in our 2007 SIAM book:





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- ▶ On the other hand, Maple, Mathematica, . . . , are very useful for symbolic calculations and multiple-precision numerics for mathematical functions.
- ▶ But be careful when using these systems when the parameters of the functions are large ...



An example

The function (Kummer, confluent hypergeometric)

$$U(a, c, z) = \frac{1}{\Gamma(a)} \int_0^\infty t^{a-1} (1+t)^{c-a-1} e^{-zt} dt,$$

is positive for real values of c and positive values of a and z .

If $c - a - 1 < 0$ then

$$U(a, c, z) < z^{-a}.$$



An example

Using Maple 16 we compute this function, taking

$$\text{Digits} = 20, \quad z = 500, \quad c = 2.5,$$

and several values of a . Results:

a	$U(a, c, z)$	z^{-a}
50.00	0.	0.1126×10^{-134}
50.20	0.	0.3250×10^{-135}
50.40	-0.1257×10^{108}	0.9375×10^{-136}
50.60	-0.1257×10^{110}	0.2705×10^{-136}
50.80	0.3142×10^{109}	0.7805×10^{-137}
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In numerical algorithms, asymptotic expansions should be used for large parameters.



Introducing the problem

Why using asymptotic expansions to compute SFs?

Building efficient algorithms for special functions may become problematic when several large parameters are involved. In particular, problems arise when functions suddenly change their behavior, say from monotonic to oscillatory behavior. For many special functions of mathematical physics, powerful uniform asymptotic expansions are available which describe precisely how the functions behave, which are valid for large domains of the parameters, and which provide tools for designing high-performance computational algorithms.



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Let's consider the case of Fermi-Dirac integrals.



Introducing the problem

We use the following notations. The classical nonrelativistic Fermi-Dirac integral is defined as

$$F_q(\eta) = \int_0^\infty \frac{x^q}{e^{x-\eta} + 1} dx, \quad q \geq 0, \quad \eta \in \mathbb{R}, \quad (1)$$

and the relativistic integral is

$$F_q(\eta, \beta) = \int_0^\infty \frac{x^q \sqrt{1 + \beta x/2}}{e^{x-\eta} + 1} dx, \quad \beta \geq 0, \quad q \geq 0, \quad \eta \in \mathbb{R}. \quad (2)$$



Introducing the problem

For finding asymptotic expansions for the relativistic FD integral, the **confluent hypergeometric functions** $U(a, b, z)$ and ${}_1F_1\left(\begin{matrix} a \\ b \end{matrix}; z\right)$ will be a key ingredient in our analysis.



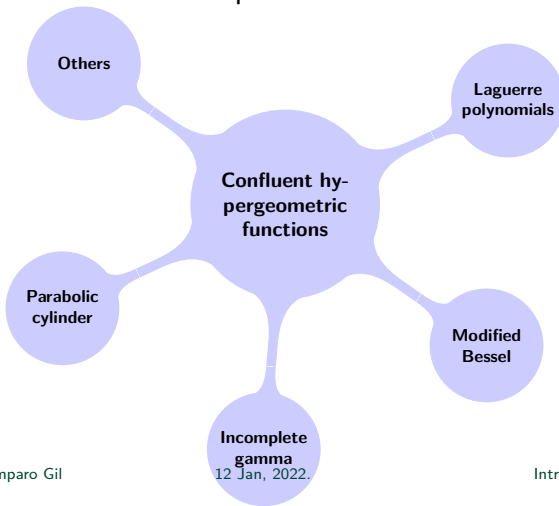
Introducing the problem

Confluent hypergeometric functions include, as particular cases, other well known special functions.



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Introducing the problem

We are working now on an algorithm to compute the CHFs

$U(a, b, z)$ and ${}_1F_1\left(\begin{matrix} a \\ b \end{matrix}; z\right)$. **This will be very useful to compute the asymptotic approximations for the relativistic FD integral.**



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Next follows an overview of our results.



Expansions for $F_q(\eta, \beta)$

Expansions in terms of η

Convergent expansions for $\eta < 0$
(all q), $\eta > 0$
($q = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$)

Expansion for large
values of η ,
 $q \neq \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$

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Expansions in terms of β

Expansion for $\beta \rightarrow 0$

Expansion for large
values of β ,
 $q \neq \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$

Expansion for large
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A first expansion for $\eta < 0$

For $\eta < 0$ we obtain

$$F_q(\eta, \beta) = \sum_{n=1}^{\infty} (-1)^{n-1} e^{n\eta} \int_0^{\infty} e^{-nx} x^q \sqrt{1 + \beta x/2} dx, \quad \eta < 0. \quad (3)$$

Then, if we use the integral representation of the Kummer $U(a, b, z)$ function

$$U(a, b, z) = \frac{1}{\Gamma(a)} \int_0^{\infty} e^{-zt} t^{a-1} (1+t)^{b-a-1} dt, \quad \Re a > 0, \quad \Re z > 0, \quad (4)$$

we obtain for $\eta < 0$ the following result



A first expansion for $\eta < 0$

Lemma

We have the convergent expansion

$$F_q(\eta, \beta) = \Gamma(q+1) \sum_{n=1}^{\infty} (-1)^{n-1} \frac{e^{n\eta}}{n^{q+1}} U_q(n, \beta), \quad \eta < 0, \quad (5)$$

in which $U_q(n, \beta)$ is a slowly varying function for large values of n .

$$U_q(s, \beta) = \left(\frac{2s}{\beta}\right)^{q+1} U\left(q+1, q+\frac{5}{2}, \frac{2s}{\beta}\right) = \left(\frac{2s}{\beta}\right)^{-\frac{1}{2}} U\left(-\frac{1}{2}, -q-\frac{1}{2}, \frac{2s}{\beta}\right), \quad (6)$$

where $s \neq 0$, $|\text{ph } s| < \pi$, $\beta > 0$, with limiting value $U_q(s, 0) = 1$.



A first expansion for $\eta < 0$

For the standard Fermi-Dirac integral the expansion in (5) becomes the well-known result

$$F_q(\eta) = \Gamma(q+1) \sum_{n=1}^{\infty} (-1)^{n-1} \frac{e^{n\eta}}{n^{q+1}}, \quad \eta < 0. \quad (7)$$



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This convergent expansion can be (of course) useful for computation ...



Numerical examples

Examples of the computational performance of (7) to approximate the non-relativistic FD integral

η	q	n	Rel. error	CPU time(s)
-6.2	3/2	7	4.4×10^{-16}	0.08×10^{-4} , (5.6×10^{-4} , num integ)
-20.3	3/2	4	4.4×10^{-16}	0.08×10^{-4} , (5.5×10^{-4} , num integ)
-100.3	3/2	3	2.1×10^{-14}	0.07×10^{-4} , (5.8×10^{-4} , num integ)



A first expansion for $\eta < 0$

The expansion given in (5) can be used for numerical computations provided we have an algorithm to compute the Kummer functions $U(a, b, z)$. For large values of z we can use the asymptotic expansion of the Kummer function

$$U(a, b, z) \sim z^{-a} \sum_{n=0}^{\infty} \frac{(a)_n (a - b + 1)_n}{n!} (-z)^{-n}, \quad |\text{ph } z| \leq \frac{3}{2}\pi - \delta, \quad (8)$$

where δ is a small positive number.



Numerical examples

Examples of the computational performance of (5) to approximate the relativistic FD integral

η	q	β	n	Rel. error	CPU time(s)
-6.2	0.8	5.2	6	2.2×10^{-14}	0.05×10^{-3} , (0.6×10^{-3} , num integ)
-20.3	0.8	5.2	3	2.1×10^{-14}	0.04×10^{-3} , (0.6×10^{-3} , num integ)
-100.3	0.8	5.2	2	2.1×10^{-14}	0.04×10^{-3} , (0.6×10^{-3} , num integ)

In the calculations, we use our algorithm to compute $U(a, b, z)$.



Numerical examples

Examples of the computational performance of (5) to approximate the relativistic FD integral

η	q	β	n	Rel. error	CPU time(s)
-6.2	3/2	5.2	6	3.1×10^{-15}	0.05×10^{-3} , (0.6×10^{-3} , num integ)
-20.3	3/2	5.2	3	2.9×10^{-15}	0.04×10^{-3} , (0.6×10^{-3} , num integ)
-100.3	3/2	5.2	2	2.9×10^{-15}	0.04×10^{-3} , (0.6×10^{-3} , num integ)

In the calculations, we use our algorithm to compute $U(a, b, z)$.



Expansions for large values of η

For fixed β and $q \neq \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$ it is possible to show that

$$F_q(\eta, \beta) = \left(\frac{2}{\beta}\right)^{q+1} \frac{\Gamma\left(-q - \frac{3}{2}\right) \Gamma(q+1)}{\Gamma\left(-\frac{1}{2}\right)} F_q^{(1)}(\eta, \beta) + \left(\frac{2}{\beta}\right)^{-\frac{1}{2}} \Gamma\left(q + \frac{3}{2}\right) F_q^{(2)}(\eta, \beta), \quad (9)$$

$$F_q^{(1)}(\eta, \beta) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\eta s} f_1(s) \frac{ds}{s},$$

$$F_q^{(2)}(\eta, \beta) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\eta s} f_2(s) \frac{ds}{s^{q+\frac{5}{2}}}.$$

where



Expansions for large values of η

$$\begin{aligned} f_1(s) &= \frac{\pi s}{\sin(\pi s)} {}_1F_1 \left(q + 1; \frac{2s}{\beta} \right), \\ f_2(s) &= \frac{\pi s}{\sin(\pi s)} {}_1F_1 \left(-\frac{1}{2}; \frac{2s}{\beta} \right), \end{aligned} \tag{10}$$



Expansions for large values of η

It is possible to obtain asymptotic expansions for $F_q^{(1)}(\eta, \beta)$ and $F_q^{(2)}(\eta, \beta)$. For fixed β and q , $q \neq \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$ we have the expansion

$$F_q^{(1)}(\eta, \beta) = \sum_{n=0}^{\infty} (-1)^n e^{-n\eta} {}_1F_1 \left(q + \frac{1}{2}; \frac{-2n}{\beta} \right), \quad (11)$$

which converges for all $\eta > 0$, and

$$F_q^{(2)}(\eta, \beta) \sim \eta^{q+\frac{3}{2}} \sum_{n=0}^{\infty} \frac{1}{\Gamma \left(q + \frac{5}{2} - n \right)} \frac{a_n}{\eta^n} + \sin(\pi q) \sum_{n=1}^{\infty} (-1)^n \frac{e^{-n\eta}}{n^{q+1}} {}_1F_1 \left(-\frac{1}{2}; \frac{-2n}{\beta} \right), \quad (12)$$

where the first series is an asymptotic expansion for $\eta \rightarrow +\infty$ and the second one converges for all $\eta > 0$.



Expansions for large values of η

The coefficients a_n are defined by the expansion

$$f_2(s) = \sum_{n=0}^{\infty} a_n s^n, \quad |s| < 1, \quad (13)$$

with $f_2(s)$ defined in (10). The first coefficients are

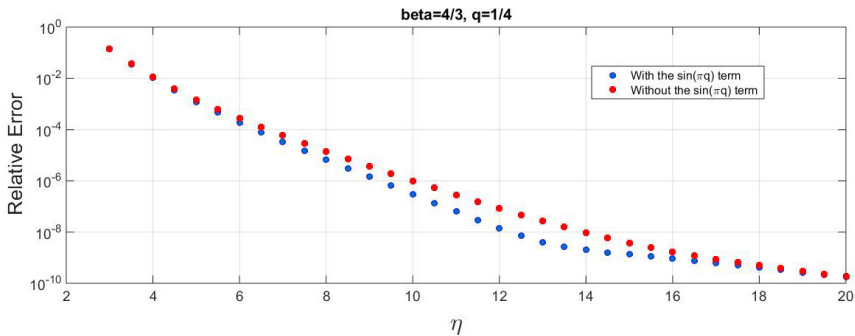
$$\begin{aligned} a_0 &= 1, \quad a_1 = \frac{2}{\beta(1+2q)}, \quad a_2 = \frac{\tau_2(\beta^2(4q^2-1)-2)}{\beta^2(1-4q^2)}, \\ a_3 &= 2 \frac{\beta^2 \tau_2(2q-1)(2q-3) + 2}{\beta^3(4q^2-1)(2q-3)}, \end{aligned} \quad (14)$$

where the coefficient τ_2 is obtained from

$$\tau_{2n} = 2 \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m^{2n}} = (-1)^{n-1} \left(1 - 2^{1-2n}\right) \frac{(2\pi)^{2n}}{(2n)!} B_{2n}. \quad (15)$$



Numerical performance





Numerical examples

Examples of the computational performance of (9) with the expansions (11) and (12) to approximate the relativistic FD integral

η	q	β	Rel. error
10	0.35	5.2	7.6×10^{-8}
30	0.35	5.2	1.9×10^{-13}
50	0.35	5.2	1.3×10^{-15}
100	0.35	5.2	7.8×10^{-16}

In the calculations, we use $n = 10$ in the asymptotic expansion of (12).



Expansions for large values of η

For $q = m - \frac{3}{2}$, $m = 2, 3, 4, \dots$ we have

$$F_q(\eta, \beta) = \Gamma\left(m - \frac{1}{2}\right) \left(\frac{2}{\beta}\right)^{m-\frac{1}{2}} \left(F_q^{(P)}(\eta, \beta) + F_q^{(Q)}(\eta, \beta)\right) + F_q^{(R)}(\eta, \beta). \quad (16)$$

For fixed β and q , we have for $\eta \rightarrow +\infty$ the asymptotic expansions

$$\begin{aligned} F_q^{(P)}(\eta, \beta) &\sim A_m \left(-(\gamma + \ln \eta) p_{m,0} + \sum_{k=1}^{\infty} (-1)^k p_{m,k} (k-1)! \eta^{-k} \right), \\ F_q^{(Q)}(\eta, \beta) &\sim A_m q_{m,0}, \end{aligned} \quad (17)$$

where γ is Euler's constant and $A_m = \frac{(-1)^{m+1}}{m! \Gamma(-\frac{1}{2})}$.



Expansions for large values of η

The coefficients $p_{m,k}$ and $q_{m,k}$ are obtained from Cauchy products

$$p_{m,k} = \sum_{j=0}^k \tau_j P_{m,k-j}, \quad q_{m,k} = \sum_{j=0}^k \tau_j Q_{m,k-j}, \quad (18)$$

where $\tau_0 = 1$, $\tau_2 = \frac{1}{6}\pi^2$, $\tau_4 = \frac{7}{360}\pi^4$, and in general

$\tau_{2n} = (-1)^{n-1} (1 - 2^{1-2n}) \frac{(2\pi)^{2n}}{(2n)!} B_{2n}$, for $n \geq 1$, where B_n are the Bernoulli numbers. Also, we have

$$P_{m,k} = \left(\frac{2}{\beta}\right)^k \frac{\left(m - \frac{1}{2}\right)_k}{k! (m+1)_k},$$

$$Q_{m,k} = P_{m,k} \left(\ln \left(\frac{2}{\beta}\right) + \psi \left(m - \frac{1}{2} + k\right) - \psi(1+k) - \psi(m+k+1) \right),$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$. (19)



Expansions for large values of η

Examples of values for the coefficients $p_{m,k}$ and $q_{m,0}$.
 $q = m - \frac{3}{2}$.

m	k	$p_{m,k}$	$q_{m,0}$
2	1	0.1886792452830189	-1.283638336216489
2	2	1.667183977848583	-1.283638336216489
2	3	0.3123240299336591e	-1.283638336216489
3	1	0.2358490566037736	-0.9503050028831552
3	2	1.676083942248725	-0.9503050028831552
3	3	0.3908948153321760	-0.9503050028831552
4	1	0.2641509433962265	-0.8003050028831553
4	2	1.682313917328825	-0.8003050028831553
4	3	0.4382052104278927	-0.8003050028831553



Expansions for large values of η

As for $F_q^{(R)}(\eta, \beta)$ we have

$$\begin{aligned}
 F_q^{(R)}(\eta, \beta) &= \Gamma\left(m - \frac{1}{2}\right) \left(\frac{2}{\beta}\right)^{m-\frac{1}{2}} \sum_{k=1}^m \frac{R_{m,k}}{\Gamma(k)} F_{k-1}(\eta) \\
 &= \sum_{j=0}^{m-1} (-1)^j \frac{\left(-\frac{1}{2}\right)_j}{j!} \left(\frac{2}{\beta}\right)^{j-\frac{1}{2}} F_{m-j-1}(\eta).
 \end{aligned} \tag{20}$$

which gives a relation in terms of the standard Fermi-Dirac integral.

The coefficients $R_{m,k}$ are given by

$$R_{m,k} = \left(\frac{2}{\beta}\right)^{-k} \frac{(k-1)! \left(\frac{3}{2} - m + k\right)_{m-k}}{\Gamma\left(m - \frac{1}{2}\right) (m-k)!}.$$



Numerical performance

Examples of the computational performance of (16) to approximate the relativistic FD integral



Expansions for large values of η

We also have the following convergent expansion for all $\eta > 0$

$$F_q(\eta, \beta) = \frac{1}{2} \left(\frac{2}{\beta} \right)^{m-\frac{1}{2}} (-1)^m \Gamma\left(m - \frac{1}{2}\right) \sum_{n=1}^{\infty} (-1)^{n-1} e^{-n\eta} \times \\ \left(U\left(m - \frac{1}{2}, m+1, \frac{2ne^{-\pi i}}{\beta}\right) + U\left(m - \frac{1}{2}, m+1, \frac{2ne^{+\pi i}}{\beta}\right) \right) \quad (21)$$



Expansions with respect to β

For fixed η and q , we have a first expansion

$$F_q(\eta, \beta) \sim \sum_{n=0}^{\infty} (-1)^n \frac{\left(-\frac{1}{2}\right)_n}{n!} \left(\frac{1}{2}\beta\right)^n F_{q+n}(\eta), \quad \beta \rightarrow 0, \quad (22)$$

where $F_q(\eta)$ is the standard Fermi-Dirac integral



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$$F_q(\eta, \beta) \sim \sum_{n=0}^{\infty} (-1)^n \frac{\left(-\frac{1}{2}\right)_n}{n!} \left(\frac{1}{2}\beta\right)^n F_{q+n}(\eta), \quad \beta \rightarrow 0, \quad (22)$$

where $F_q(\eta)$ is the standard Fermi-Dirac integral

For large values of β , we have obtained

$$F_q(\eta, \beta) = \left(\frac{2}{\beta}\right)^{q+1} \frac{\Gamma\left(-q - \frac{3}{2}\right) \Gamma(q+1)}{\Gamma\left(-\frac{1}{2}\right)} F_q^{(1)}(\eta, \beta) + \left(\frac{2}{\beta}\right)^{-\frac{1}{2}} \Gamma\left(q + \frac{3}{2}\right) F_q^{(2)}(\eta, \beta), \quad (23)$$



Expansions with respect to β

For fixed η and q , with $q \neq \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$, we have

$$F_q^{(1)}(\eta, \beta) \sim \sum_{k=0}^{\infty} \frac{c_k}{\beta^k} \Phi_k^{(1)}(\eta), \quad F_q^{(2)}(\eta, \beta) \sim \sum_{k=0}^{\infty} \frac{d_k}{\beta^k} \Phi_k^{(2)}(\eta, q),$$

$$c_k = \frac{2^k (q+1)_k}{k! \left(q + \frac{5}{2}\right)_k}, \quad d_k = \frac{2^k \left(-\frac{1}{2}\right)_k}{k! \left(-q - \frac{1}{2}\right)_k},$$
(24)

as $\beta \rightarrow \infty$.



Expansions with respect to β

For $q = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$ we write $F_q(\eta, \beta)$ as in (16):

$$F_q(\eta, \beta) = \Gamma\left(m - \frac{1}{2}\right) \left(\frac{2}{\beta}\right)^{m-\frac{1}{2}} \left(F_q^{(P)}(\eta, \beta) + F_q^{(Q)}(\eta, \beta)\right) + F_q^{(R)}(\eta, \beta), \quad (25)$$

For fixed η and q , with $q = m - \frac{3}{2}$, $m = 2, 3, 4, \dots$ we have as $\beta \rightarrow \infty$

$$F_q^{(P)}(\eta, \beta) \sim A_m \sum_{k=0}^{\infty} \frac{\tilde{P}_{m,k}}{\beta^k} \Psi_k(\eta), \quad F_q^{(Q)}(\eta, \beta) \sim A_m \sum_{k=0}^{\infty} \frac{\tilde{Q}_{m,k}}{\beta^k} \Phi_k^{(1)}(\eta), \quad (26)$$



Expansions with respect to β

$F_q^{(R)}(\eta, \beta)$ is the same as in (20) and

$$\begin{aligned}\tilde{P}_{m,k} &= \frac{2^k (m - \frac{1}{2})_k}{k! (m+1)_k}, \quad A_m = \frac{(-1)^{m+1}}{m! \Gamma(-\frac{1}{2})}, \\ \tilde{Q}_{m,k} &= \tilde{P}_{m,k} \left(\ln \left(\frac{2}{\beta} \right) + \psi \left(m - \frac{1}{2} + k \right) - \psi(1+k) - \psi(m+k+1) \right).\end{aligned}\tag{27}$$



Expansions with respect to β

The auxiliary functions $\Phi_k^{(1)}(\eta)$, $\Phi_k^{(2)}(\eta, q)$ and $\Psi_k(\eta)$ appearing in the expansions should be considered with care.



Expansions with respect to β

We have

$$\begin{aligned}\Phi_k^{(1)}(\eta) &= \hat{F}_{-k-1}(\eta), \quad \Phi_k^{(2)}(\eta, q) = \hat{F}_{q+\frac{1}{2}-k}(\eta), \\ \Psi_k(\eta) &= - \left. \frac{\partial}{\partial q} \hat{F}_q(\eta) \right|_{q=-k-1}, \quad k = 0, 1, 2, 3, \dots, \end{aligned} \tag{28}$$

where

$$\begin{aligned}\hat{F}_q(\eta) &= \frac{F_q(\eta)}{\Gamma(q+1)} = \frac{1}{\Gamma(q+1)} \int_0^\infty \frac{x^q}{e^{x-\eta} + 1} dx \\ &= \frac{1}{2i} \int_{c-i\infty}^{c+i\infty} \frac{e^{\eta s}}{s^{q+1} \sin(\pi s)} ds, \quad 0 < c < 1. \end{aligned} \tag{29}$$



Expansions with respect to β

Comments:

1. By analytical continuation it can be shown that the function $\hat{F}_q(\eta)$ is well defined for all complex values of q .
2. The numerical evaluation of $\hat{F}_q(\eta)$ for $q \leq -1$ is made as follows:

We write

$$\begin{aligned}\hat{F}_q(\eta) &= \hat{F}_q^{(1)}(\eta) + \hat{F}_q^{(2)}(\eta), \\ \hat{F}_q^{(1)}(\eta) &= \frac{1}{\Gamma(q+1)} \int_0^1 \frac{x^q}{e^{x-\eta} + 1} dx, \\ \hat{F}_q^{(2)}(\eta) &= \frac{1}{\Gamma(q+1)} \int_1^\infty \frac{x^q}{e^{x-\eta} + 1} dx.\end{aligned}\tag{30}$$



Expansions with respect to β

$\widehat{F}_q^{(2)}(\eta)$ can be computed by numerical quadrature for all complex q , and vanishes for $q = -1, -2, -3, \dots$

For $\widehat{F}_q^{(1)}(\eta)$ it is possible to obtain

$$\widehat{F}_q^{(1)}(\eta) = \frac{\Gamma(-q)}{2\pi} \int_{-\pi}^{\pi} f(\theta) d\theta = \frac{\Gamma(-q)}{\pi} \int_0^{\pi} f(\theta) d\theta, \quad (31)$$

where

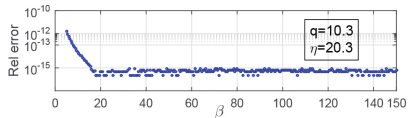
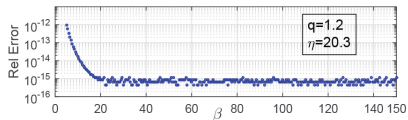
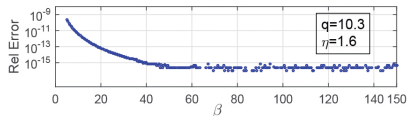
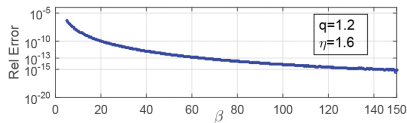
$$f(\theta) = \frac{e^{-\eta - \cos(\theta)} \cos(\mu\theta + \sin(\theta)) + \cos(\mu\theta)}{1 + 2e^{-\eta - \cos(\theta)} \cos(\sin(\theta)) + e^{-2\eta - 2\cos(\theta)}}, \quad \mu = q + 1. \quad (32)$$

This representation is quite convenient for large positive values of η (for large negative values we can multiply the numerator and denominator by $e^{2\eta}$).



Numerical performance

Examples of the computational performance of (23) to approximate the relativistic FD integral





Numerical performance

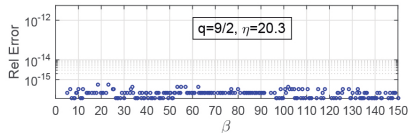
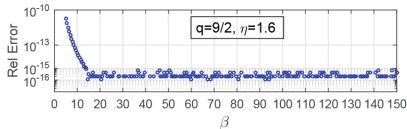
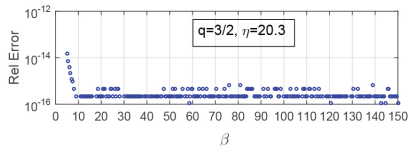
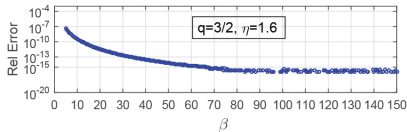
Examples of the computational performance of (23) to approximate the relativistic FD integral. We take $q = 2.4$, $\eta = 9/2$.

k	$\beta = 50$	$\beta = 100$
0	4.2×10^{-3}	2.1×10^{-3}
1	1.1×10^{-5}	2.8×10^{-5}
2	9.8×10^{-8}	1.2×10^{-8}
3	4.7×10^{-9}	2.9×10^{-10}
4	1.7×10^{-12}	5.3×10^{-14}
5	8.5×10^{-15}	2.2×10^{-16}



Numerical performance

Examples of the computational performance of (25) to approximate the relativistic FD integral





For further details

For more information see

Complete asymptotic expansions for the relativistic Fermi-Dirac integral. A. Gil, J. Segura, N. M. Temme. Appl. Math. Comput. 412 (2022) 126618

<https://arxiv.org/abs/2108.11210>



Further work

Expansions for the derivatives (with respect to the parameters) of the FD relativistic integral are also interesting. For example:

$$\frac{\partial F_q}{\partial \eta}(\eta, \beta) = \int_0^\infty \frac{x^q \sqrt{1 + \beta x/2}}{(e^{(x-\eta)/2} + e^{(\eta-x)/2})^2} dx, \quad \beta \geq 0, \quad q \geq 0, \quad \eta \in \mathbb{R}. \quad (33)$$

Taking the derivative of (5) with respect to η , we obtain

$$\frac{\partial F_q}{\partial \eta}(\eta, \beta) = \Gamma(q+1) \sum_{n=1}^{\infty} (-1)^{n-1} \frac{e^{n\eta}}{n^q} U_q(n, \beta), \quad \eta < 0. \quad (34)$$



Numerical examples

Examples of the computational performance of (5) to approximate the relativistic FD integral

η	q	β	n	Rel. error	CPU time(s)
-6.2	0.8	5.2	6	2.2×10^{-14}	0.05×10^{-3} , (0.6×10^{-3} , num integ)
-20.3	0.8	5.2	3	2.1×10^{-14}	0.04×10^{-3} , (0.6×10^{-3} , num integ)
-100.3	0.8	5.2	2	2.1×10^{-14}	0.04×10^{-3} , (0.7×10^{-3} , num integ)



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Thank you!