Excercise 1.

„Derive" 2-point discrete derivative formula

$$f'(x_k) = \frac{f(x_{k+1}) - f(x_k)}{\Delta x}.$$

Please code above formula in your favourite programming language, and check precision and accuracy as a function of Δx for some well-known elementary function. Value of Δx should be decreased geometrically, e.g., $\Delta x = 2^{-n}, \quad n = 0 \dots 64$.

What is happening if you decrease Δx down to *machine epsilon*, $\epsilon = 2^{-53}$ for `double` in C? What is optimal value of Δx and why?

Redo analysis using higher-order approximation for first and second derivative.

Excercise 2.

Derive and check formulas for $f'(x)$ and $f''(x)$ using *stencil*:

2.1 X–X–fX–X–X,

2.2 X–X–fX–X,

2.3 fX–X–X–X,

2.3 f–X–X–X,

2.4 fX–X–X.

Symbol fX denotes grid point where derivative is to be evaluated, and X neighboring grid points used to compute derivative. Single letter f denotes grid point which is not used to compute derivative, e.g, it is beyond area covered with numerical grid. Assume constant distance Δx between grid points.

Excercise 3.

Find first and second discrete derivative formulas for geometrically spaced grid, i.e., $x_k = x_0 q^k$. A *stencil* has a form:

3.1 X–fX–X,

3.2 f–X–X,

3.3 fX–X–X.

Excercise 4.

Derive and check **Laplace operator** Δ in 2D:

$$\Delta f(x, y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.$$

using 5-point *stencil*:
O—X—O

X–fX—X
O—X—O
where O denotes grid points unused in computation.

Excercise 5.

Find disrete formulas for partial derivatives:

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \quad \frac{\partial^2 f}{\partial x \partial y}, \quad \frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y^2},$$

in a corner of the rectangular grid:
fX–X—X
X—X—O
X—O—O

Excercise 6.

Find formulas for Laplace operator and mixed partial derivatives (see Excercises 4 & 5), using grid composed of equilateral triangles with side length Δx. Assume *stencil* in the form of hexagonal vertices, and compute derivatives in the center of hexagon.