Elastic Planetoids

Analytical model for self-gravitating solid and liquid bodies

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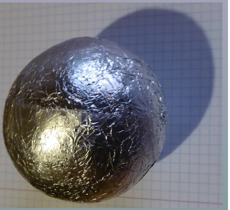
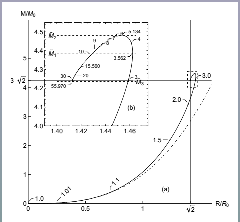
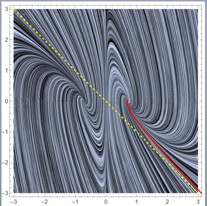
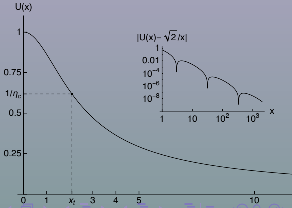
21 Jan 2026









A simple question to start...

Iron vs. Aluminum

- We have two balls: one iron, one aluminum
- Which one would be **denser** on average if scaled up to asteroid size?

The key insight

Self-gravitating bodies compress under their own weight!

The answer depends on *both*
bulk modulus K and uncompressed density ρ_0.

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Motivation: Why another model? (1)

1. Artificial "planetoid" for G measurements

- Deep space experiments require strong source of gravity
- Can we manufacture objects *much denser* than osmium?

2. Interstellar objects with exotic composition

- 1I/'Oumuamua (N iceberg), 2I/Borisov, 3I/ATLAS (pure Ni)
- Hypatia stone: highly anomalous presolar material (Ti grains)
- Helix nebula ("cometary" balls)
- Overdense asteroids (Polyhymnia, $\bar{\rho} = 12.4\text{-}75\ \mathrm{g\ cm^{-3}}$) and exoplanets (PSR J1719-1438 b $\bar{\rho} > 23\ \mathrm{g\ cm^{-3}}$)

Need flexible model with just ρ_0 and K parameters

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Need flexible model with just ρ_0 and K parameters

Motivation: Why another model? (2)

3. Teaching in the age of AI

- Lane-Emden, Chandra models: already in LLM training data
- Students can get instant solutions from ChatGPT/Claude
- **Fresh problem:** Elastic model is new!
- Visualizable: liquid bodies, iron spheres, asteroids

4. "Bridge" theory for small Solar System bodies

- Constant density: too simplistic
- **Elastic model:** tractable middle ground!
- Numerical EOS: too complex for analytical insight

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The Elastic Equation of State

Starting from bulk modulus definition

$$K = \rho \frac{dP}{d\rho}$$

Integration yields the **logotropic EOS**:

$$P(\rho) = K \ln\left(\rho/\rho_0\right) = K \ln\left(\eta\right) = K \ln\left(1+\mu\right)$$

Valid for $\rho \geq \rho_0$ (compression only).

The $\eta = \rho/\rho_0$ is compression, $\mu = \eta - 1$ is dilatation.

Two parameters only!

- ρ_0 = uncompressed density [g cm^{-3}]
- K = bulk modulus [GPa]

Derivation of the Elastic ODE

Hydrostatic equilibrium

$$h + \Phi_g = C(onst), \quad \text{where } h = \int \frac{dP}{\rho}, \; \Delta\Phi_g = 4\pi G\rho$$

Taking spherical Laplacian $\Rightarrow$

$$\rho'' + \frac{2\rho'}{r} - \frac{2(\rho')^2}{\rho} + \frac{4\pi G}{K}\rho^3 = 0$$

Dimensionless form

Substituting $\rho(r) = \rho_c u(r/\lambda)$ with scale:

$$\lambda^2 = \frac{K}{4\pi G\rho_c^2}$$

The Elastic ODE

Dimensionless equation

$$\boxed{u'' + \frac{2}{x}u' - \frac{2u'^2}{u} + u^3 = 0}$$

Comparison with Lane-Emden

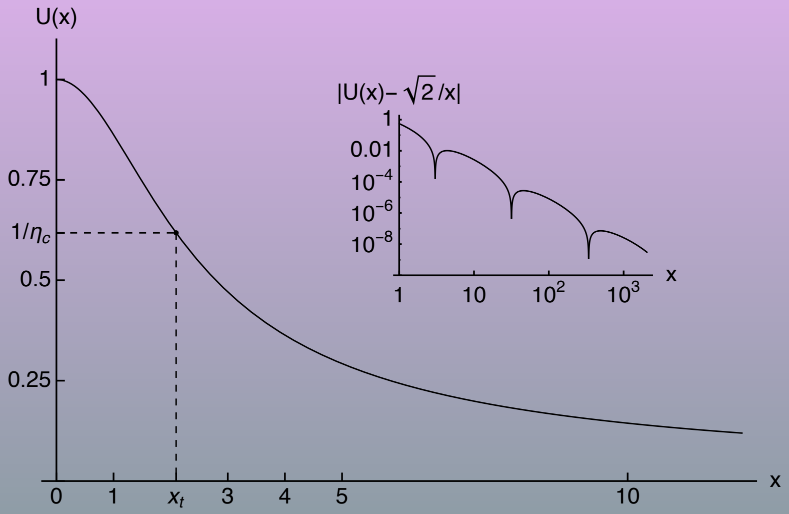
- Lane-Emden ($n = 3$): $\theta'' + \frac{2}{x}\theta' + \theta^3 = 0$
- **Extra term:** $-2u'^2/u$
- Alternative form:
  $\Delta(1/u) - u = 0,\ \Delta(w) = 1/w,\ w(x) \equiv 1/u(x)$

Initial conditions

$$u(0) = 1, \quad u'(0) = 0$$

define special function $U(x)$, denoted by capital U to distinguish it from general elastic ODE solutions, $u(x)$.

The Universal Elastic Function $U(x)$



Properties of $U(x)$: Series expansion

Taylor series at $x = 0$

$$U(x) \simeq 1 - \frac{1}{6}x^2 + \frac{13}{360}x^4 - \frac{25}{3024}x^6 + \dots$$

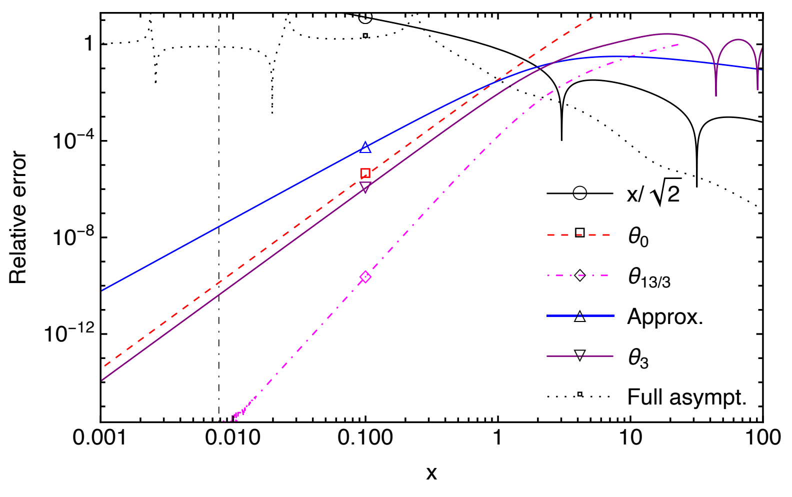
- Matches Lane-Emden with $n \simeq 13/3 \simeq 4.33$
- Diverges starting from x^6 term
- Small convergence radius (like Lane-Emden)

Asymptotic behavior

$$U(x) \sim \frac{\sqrt{2}}{x} \quad \text{as } x \to \infty$$

This is the **exact singular solution**!

Approximations to $U(x)$



The Hidden Oscillation

Oscillatory asymptotics

$$U(x) \simeq \frac{\sqrt{2}}{x} + A\frac{\cos\left(\frac{\sqrt{7}}{2}\ln x + \phi_0\right)}{\sqrt{x}}$$

where $A \simeq 0.4164$, $\phi_0 \simeq 0.1395$ (from numerics).

Physical interpretation: damped oscillator!

Changing $x = e^t$, $W(t) \equiv w(e^t) = e^t z(t)$ leads to:

$$\frac{1}{2}v^2 + V(z) = E, \quad \dot{E} = -3v^2, \; v = \dot{z}$$

with potential $V(z) = z^2 - \ln z$. $V_{min}(1/\sqrt{2}) = \ln\sqrt{2e}$

Damping coefficient 3 is close to critical value 4, above which oscillations vanish.

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Existence of elastic function $U(x)$ (with P.Bizoń & P.Mach)

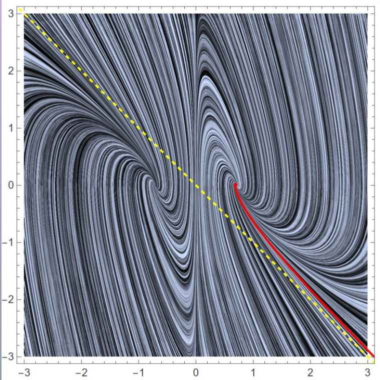
Initial value problem:

$$\Delta w \equiv \frac{1}{z^2}\frac{d}{dz}\left(z^2\frac{dw}{dz}\right) = \frac{1}{w}, \qquad w(0) = 1, w'(0) = 0.$$

1. Function $1/w$ is Lipshitz-continuous near $w = 1$.
2. Problem is equivalent to Volterra integral equation:

$$w(x) = 1 + \int_0^x \left(s - \frac{s^2}{x}\right)\frac{ds}{w(s)},$$

which can be solved uniquely using fixed-point method.



Autonomous system: $\dot{v} = 1/z - 2z - 3v, \ \dot{z} = v$

Scaling Parameters

Characteristic scales

- **Length scale:** $R_0^2 = \dfrac{K}{4\pi G \rho_0^2}$
- **Mass scale:** $M_0 = \dfrac{4}{3}\pi \rho_0 R_0^3$

Typical values (from Table 1)

- R_0: 400 km (Hg) to 7000 km (Boron) $\Rightarrow$ factor of 17
- M_0: $10^{-3} M_L$ (He) to $56 M_L$ (diamond)
- Moon mass M_L (Lunar mass) is natural unit

Dimensionless parameters

$$\tilde{R} = R/R_0, \qquad \tilde{M} = M/M_0$$

Physical Quantities from $U(x)$

Key relations

$$\text{Radius:} \quad \tilde{R} = x_t U(x_t)$$

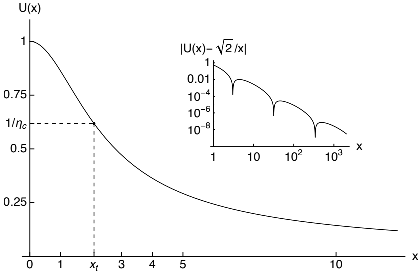
$$\text{Mass:} \quad \tilde{M} = -3x_t^2 U'(x_t)$$

$$\text{Density contrast:} \quad \frac{\rho_c}{\bar{\rho}} = -\frac{x_t}{3}\frac{U(x_t)^2}{U'(x_t)}$$

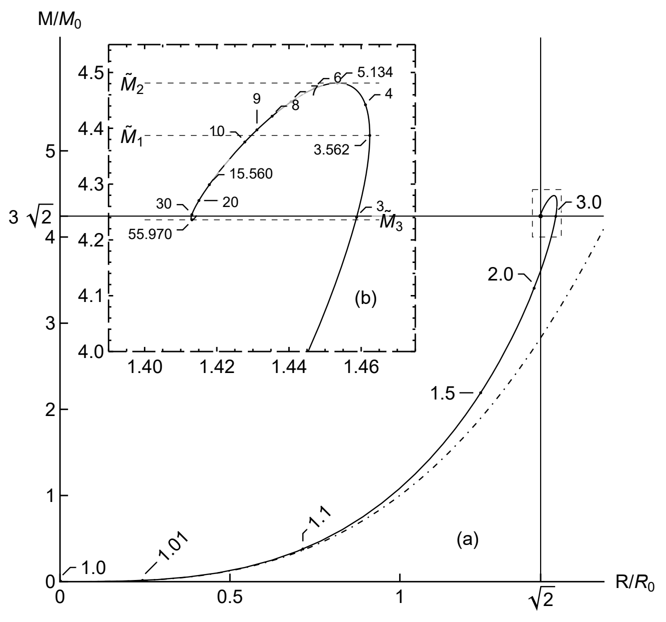
Surface condition

$$U(x_t) = 1/\eta_c = \rho_0/\rho_c$$

One-to-one mapping between x_t and central compression η_c.



The Mass-Radius Curve



Mass-Radius Curve: Key Features

Surprising termination

- Constant density: $\tilde{M} = \tilde{R}^3$ (unbounded)
- Elastic model: **terminates** at $(\sqrt{2}, 3\sqrt{2})$.

Critical points

Point	η_c	$\tilde{R}$	$\tilde{M}$
Max radius	3.56	1.462	4.387
Max mass	5.13	1.453	4.481
Secular limit	55.97	1.413	4.236
Asymptote	∞	$\sqrt{2}$	$3\sqrt{2}$

Stability Analysis

The spiraling behavior

M(R) curve spirals toward terminating point!
Distance from asymptote $\propto \eta_c^{-3/2}$

Stability thresholds

- $\tilde{M} > \tilde{M}_3 \approx 4.236$: **Secular instability**
- $\tilde{M} > \tilde{M}_2 \approx 4.481$: **Dynamical instability**

Purely Newtonian (!) instability

$R_g/R_0 = \frac{2}{3}(c_s/c)^2 \sim 10^{-10}$ for iron.
General relativity plays **no role**!

Table 1: Material Parameters

Material	Phase	ρ_0 [g/cm^3]	K [GPa]	R_0 [km]	M_0/M_L	Type
Iron	solid	7.87	159	1780	2.5	Inner SS
Quartz	solid	2.65	37	2520	2.4	Inner SS
Water ice	ice	0.92	8.8	3530	2.3	Outer SS
N_2 ice	ice	0.85	2.2	1888	0.32	Outer SS
Diamond	solid	3.52	439	6530	**56**	Max M_0
Helium	liquid	0.125	0.004	550	**0.001**	Min M_0
Boron	solid	2.36	225	**6940**	45	Max R_0
Mercury	liquid	13.5	28	**425**	0.06	Min R_0
Osmium	solid	22.6	400	970	1.2	Max ρ_0, K

Table 1: Exotic Cases

Curious examples

Material	ρ_0	K	R_0	M_0/M_L
Marshmallow	0.36	8e-5 GPa	27 km	4×10^{-7}
Strange quark	4.7×10^{14}	1.3×10^{25} GPa	8.4 km	1.6×10^{7}

The marshmallow planetoid

Self-gravitating marshmallow: $R_0 \sim 27$ km, $M_0 \sim 10^{-7} M_L$

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Universal Quadratic Profile

Simple approximation: $U(x) \approx 1 - x^2/6$

Density profile:

$$\boxed{\rho(r) = \rho_c \left(1 - \frac{r^2}{R^2}\right) + \rho_0 \frac{r^2}{R^2}}$$

Resulting formulas

- Mass: $M = \frac{4}{3}\pi R^3 \rho_0 (1 + \frac{2}{5}\mu_c)$
- Moment of inertia: $I = \frac{2}{5} M R^2 \frac{1+2\mu_c/7}{1+2\mu_c/5}$
- Compactness: $\tilde{M}/\tilde{R} = 6\mu_c$

Important Analytical Formulas

Gravitational potential at center

$$\Phi_g(0) = -\pi G(\rho_c + \rho_0)R^2$$

Binding (gravitational) energy

$$\boxed{\Omega = -\frac{16\pi^2 G R^5}{315}\left(4\rho_c^2 + 10\rho_c\rho_0 + 7\rho_0^2\right)}$$

Escape velocity from center

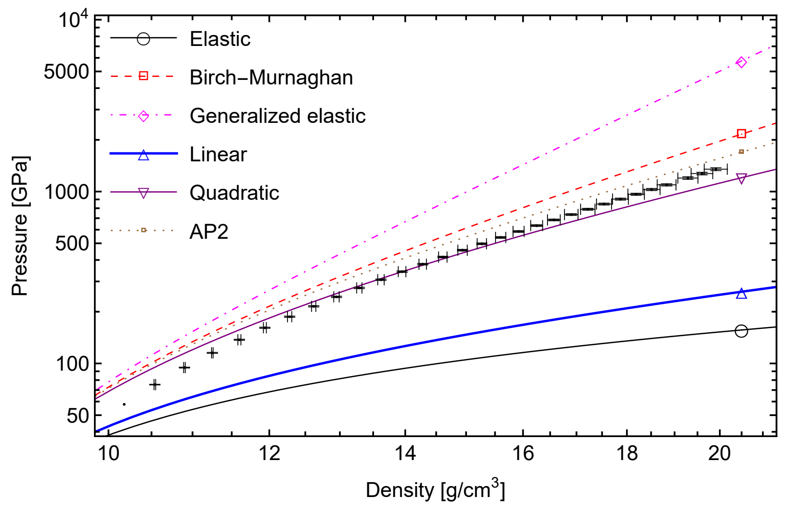
$$v_{esc}(0) = \sqrt{2|\Phi_g(0)|} = \sqrt{2\pi G(\rho_c + \rho_0)}\,R$$

Binding Energy and "Don't Look Up" scenario

Why precise binding energy matters

- Asteroid dismantling requires binding energy estimates
- Constant density model: $\Omega = -\frac{3GM^2}{5R}$
- Elastic model factor $1 + \frac{2}{21}\mu_c(9 + 2\mu_c)$
- For $\mu_c = 0.1$: correction $\sim 0.1\%$

Comparison with Real Iron EOS (NIF data)



EOS Comparison: Key Results

National Ignition Facility data (Smith et al. 2018)

High-pressure iron EOS measured up to $\eta \sim 2.5$

Analytical formulas tested

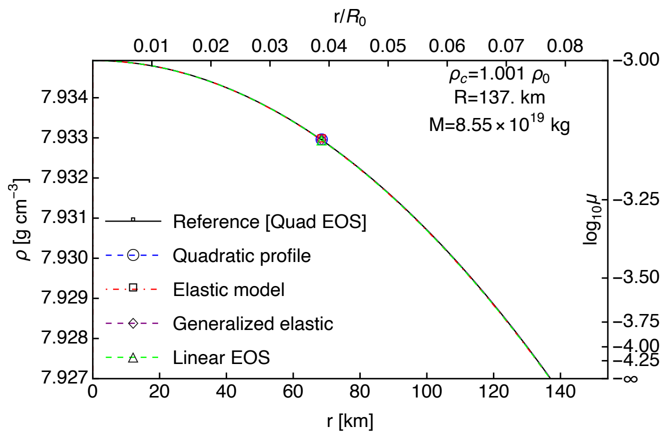
1. **AP2 (Holzapfel):** Best fit at high density
2. **Birch-Murnaghan:** Good, but complex
3. **Quadratic EOS:** Surprisingly competitive!
4. **Elastic (logotropic):** Valid for $\eta < 1.1$

Important observation

Quadratic EOS: $P = K\mu - \frac{1}{2}K(1 - K')\mu^2$
performs almost as well as sophisticated models!

CAVEAT: measuring K is students LAB exercise (F-6); measuring K' is Nature paper.

Density Profiles: Elastic vs. Reference



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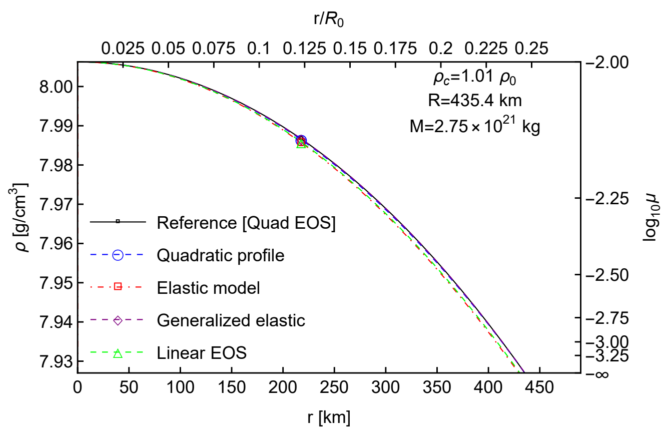

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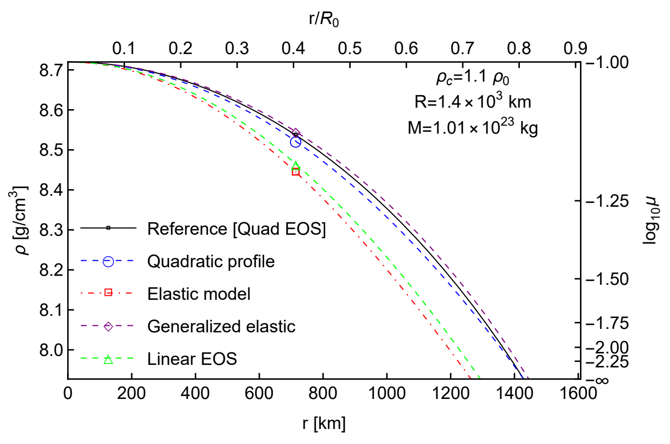

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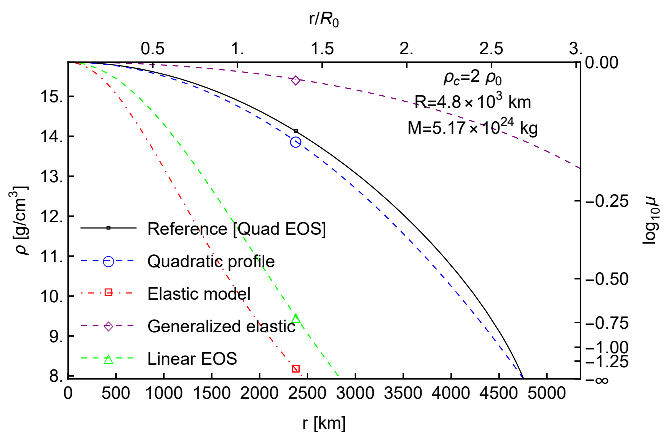


Validity Range Summary

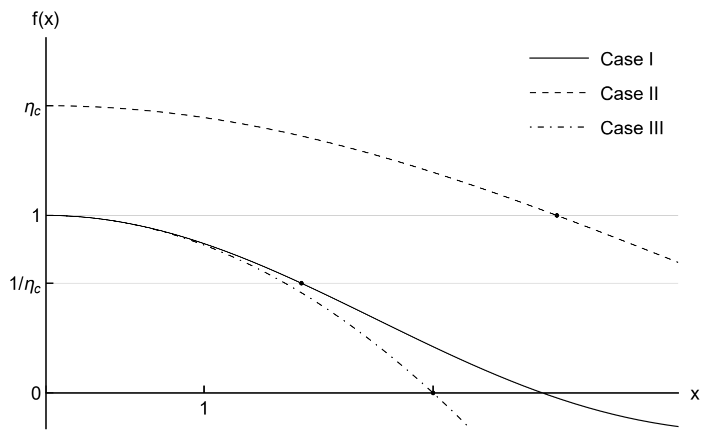
When to use elastic model

- $\tilde{R} < 1$ (radius less than scale R_0)
- $\tilde{M} < 1$ (mass less than scale M_0)
- Central compression $\mu_c < 0.1$

Examples

Object	Mass	Status
Mimas, Vesta	$\sim 10^{20}$ kg	Valid
Ceres, Pluto	$\sim 10^{21}$ kg	Marginal
Moon, Mercury	$\sim 10^{22}$ kg	Limit
Venus, Earth	$> 10^{23}$ kg	Invalid

Unified Derivation: Multiple EOSs



Family of Dimensionless ODEs

Resulting equations

$$\text{Elastic:} \quad \Delta U - \frac{2U'^2}{U} + U^3 = 0$$

$$\text{Lane-Emden:} \quad \Delta\theta + \theta^n = 0$$

$$\text{Chandrasekhar:} \quad \Delta\varphi + \left(\varphi^2 - 1/z_c^2\right)^{3/2} = 0$$

$$\text{Linear:} \quad \Delta L - \frac{L'^2}{L} + L^2 = 0$$

$$\text{Quadratic:} \quad \Delta Q + \frac{(1-\delta)Q^3 + \delta Q'^2}{Q(Q-\delta)} = 0$$

Common framework

All derived from $h + \Phi_g = const$ with appropriate scaling!

Future Prospects

Natural extensions

1. **Elastic envelopes:** Piecewise EOS (polytropic core + elastic mantle/ocean)
2. **Hollow cavity bodies:** Central cavities (space mining, micro BH scenarios)
3. **Rotating model:** Break-up limits, YORP effect for asteroids

Quadratic EOS investigation

$$\Delta Q + \frac{(1-\delta)Q^3 + \delta Q'^2}{Q(Q-\delta)} = 0$$

Surprisingly effective approximation $\Rightarrow$ deserves detailed study!

Alternative gravity follow-up

- Modified gravity theories: how does $M(R)$ change?

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Summary

Main contributions

1. New analytically tractable model for solid/liquid bodies
2. Universal function $U(x)$ encodes all physics
3. Surprising result: **maximum mass & radius exist!**
4. Oscillatory approach to termination point
5. Simple approximations for asteroid characterization
6. Unified framework connecting multiple EOS models

Take-home messages

- Self-gravitating elastic bodies have **finite maximum mass**
- Instability is purely **Newtonian** (no GR needed!)
- Average density limited to $\sim 1.46\rho_0$ (stable branch)
- Quadratic density profile is **surprisingly universal**
- Model valid for asteroids, small moons ($\tilde{R}, \tilde{M} < 1$)

Thank you!

ApJ article: doi:10.3847/1538-4357/ad9588

Extra slides

Helix Nebula *cometary knots*

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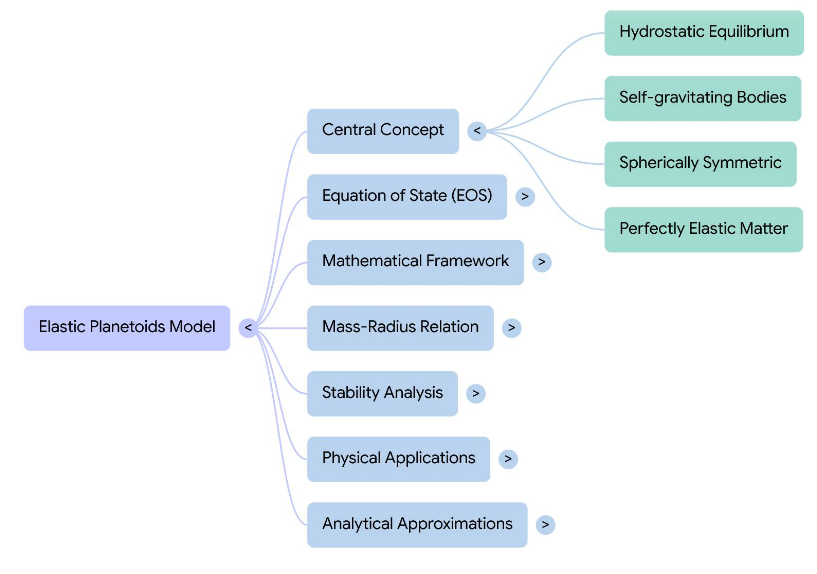
Topic landscape



Elastic ocean + polytropic core