Analysis of transverse momentum distributions observed at the RHIC by a stochastic model in the hyperbolic space

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1. Stochastic equation in longitudinal rapidity and transverse momentum variables

2. Diffusion equation in the hyperbolic space $H^3$

3. Analysis of Transverse momentum distributions observed at the RHIC

4. Summary
1. Stochastic equation in longitudinal rapidity and transverse momentum variables

Ornstein-Uhlenbeck process in 3D

\[
\frac{\partial}{\partial t} \phi(\mathbf{r}, t) = \sum_{i=1}^{2} \left\{ \mu_i \frac{\partial}{\partial x_i} x_i \phi(\mathbf{r}, t) + D_i \frac{\partial^2}{\partial x_i^2} \phi(\mathbf{r}, t) \right\} + \gamma \frac{\partial}{\partial x_3} x_3 \phi(\mathbf{r}, t) + D_3 \frac{\partial^2}{\partial x_3^2} \phi(\mathbf{r}, t)
\]

initial condition: \( \phi(\mathbf{r}, t = t_0) = \delta^3(\mathbf{r} - \mathbf{r}_0) \)

\[
\begin{align*}
\{ & \mu_1 = \mu_2 = \mu_r \\
& D_1 = D_2 = D_r \}
\end{align*}
\begin{align*}
\{ & D_3 = D \\
& x_3 = y \\
& x_{30} = y_0 \}
\end{align*}
\begin{align*}
\{ & x_1 = \rho \cos \theta \\
& x_2 = \rho \sin \theta \\
& x_{10} = \rho_0 \cos \theta_0 \\
& x_{20} = \rho_0 \sin \theta_0 \}
\end{align*}

\[
\begin{align*}
\{ & \mathbf{r}_1 = \frac{\mathbf{p}_r}{m} \\
& \mathbf{r}_2 = \frac{\mathbf{p}_r}{m} \}
\end{align*}
\begin{align*}
\{ & p_r = 1 - e^{-2\mu_r(t-t_0)} \\
& p_\gamma = 1 - e^{-2\gamma(t-t_0)} \}
\end{align*}

\[
\phi(\mathbf{r}, t) = (2\pi)^{-3/2} \frac{\mu_r}{D_r p_r} \left( \frac{\gamma}{D p_\gamma} \right)^{1/2} \exp \left\{ \frac{\gamma(y - y_0 \sqrt{1 - p_\gamma})^2}{2D p_\gamma} \right\} \left\{ \frac{\mu_r [(x_1 - x_{10} \sqrt{1 - p_r})^2 + (x_2 - x_{20} \sqrt{1 - p_r})^2]}{2D_r p_r} \right\}
\]

where \( \phi_\gamma(y, t) : \) rapidity distribution,
\( \rho \rightarrow p_T : \phi_r(\rho, \theta, t) : p_T \) distribution
After $\theta$ integration
\[
\int_0^{2\pi} d\theta \phi_r(r, t) = \frac{\mu_r}{D_r p_r} \exp \left\{ -\frac{\mu_r}{2D_r p_r} \left[ \rho^2 + \rho_0^2 (1 - p_r) \right] \right\} \times I_0 \left( \frac{\mu_r \rho_0 \sqrt{1 - p_r}}{D_r p_r} \right)
\]
\[
\rho^2 = p_T^2, \quad \rho_m = \sqrt{m_0^2 + \langle p_T \rangle^2}, \quad D_r / \mu_r = A
\]
\[
= \frac{1}{A p_r} \exp \left\{ -\frac{p_T^2 + \rho_m^2 (1 - p_r)}{2A p_r} \right\} \times I_0 \left( \frac{p_T \rho_m \sqrt{1 - p_r}}{A p_r} \right)
\]

This Equation is equivalent to the expression obtain by Volo-

schin, Phys. Rev. C55 (1997) 1630,
Transverse radial expansion and directed flow

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The effects of an interplay of radial expansion of the thermalized system created in a heavy ion collision and directed flow are discussed. It is shown that the study of azimuthal anisotropy of particle distribution as a function of rapidity and transverse momentum could reveal important information on both radial and directed flow. [S0034-6893(97)50304-9]

PACS number(s): 25.75.Ld, 24.10.Jv

The transverse isotropic expansion of the source can be described as a superposition of different sources moving radially with an expansion velocity $\beta_0$. Then the (nonrelativistic) transverse momentum distribution of protons from a radially expanding thermal source can be written as

$$ \frac{1}{N} \frac{d^2 N}{d\beta_T} = \frac{1}{(2\pi)^2(2mT)} \int d\phi \exp
$$

$$ \times \left[ \frac{\left[p_x - p_0 \cos(\phi)\right]^2 + \left[p_y - p_0 \sin(\phi)\right]^2}{2mT} \right] I_0(\xi), \quad \text{(4)} $$

where $p_0 = m\beta_0$. The integration over $\phi$ (the orientation of the expansion velocity) results in the distribution

$$ \frac{1}{N} \frac{d^2 N}{d\beta_T} = \frac{1}{2\pi(2mT)} \exp \left[ -\frac{p_x^2 + p_y^2}{2mT} / I_0(\xi) \right], \quad \text{(5)} $$

where $I_0$ is the modified Bessel function, and $\xi = \beta_0 \rho / T$.

Using the formula (3) one gets an expression for $v_1$:

$$ v_1 \approx \frac{\langle p_x \rangle}{\langle p_T \rangle} $$

FIG. 1. $v_1(p_T)$: solid lines—Eq. (6), dashed lines—Eq. (8).
$T=0.1$ GeV, $\beta_0=0.1$.

$$ v_1(p_T) = \frac{p_0 \beta_0}{2T} \frac{1 - m \beta_0}{p_T} I_0(\xi) \quad \text{(6)} $$

In our analysis we work in the frame moving longitudinally with the effective source rapidity. In this frame the longitudinal particle momenta usually can be neglected, which was done in the derivation above. If one considers the particle production with rapidities far from the rapidity of the effective source, when particle longitudinal momenta are large, one should make a substitution $m \to \sqrt{m^2 + p_T^2}.$
2. Diffusion equation in the Hyperbolic Space $H^3$

The geodesic polar coordinate

\[ v_1 = \frac{p_1}{E} = \tanh \rho \cos \theta \]
\[ v_2 = \frac{p_2}{E} = \tanh \rho \sin \theta \cos \phi \]
\[ v_3 = \frac{p_3}{E} = \tanh \rho \sin \theta \sin \phi \]

\[ E = m \cosh \rho \]
\[ p = \sqrt{p_1^2 + p_2^2 + p_3^2} = m \sinh \rho \]
\[ \rho = \ln \left( \frac{E + p}{m} \right) \]

$\rho$ : radial rapidity

The metric is written as

\[ ds^2 = d\rho^2 + \sinh^2 \rho \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \]
Diffusion equation

\[
\frac{\partial f}{\partial t} = \frac{D}{\sinh^2 \rho} \left[ \frac{\partial}{\partial \rho} \left( \sinh^2 \rho \frac{\partial f}{\partial \rho} \right) \right. \\
+ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2} \right]
\]

initial condition

\[f(\rho, \theta, \phi, t = 0) = \frac{\delta(\rho) \delta(\theta) \delta(\phi)}{\sinh^2 \rho \sin \theta}\]

Approximate solution for \(Dt << 1\)

\[f = (4\pi Dt)^{-3/2} e^{-Dt} \frac{\rho}{\sinh \rho} \exp \left[ -\frac{\rho^2}{4Dt} \right]\]

Molchanov, Russ. Math. Surveys, 30 (1975) 1-63

This is the solution of the equation with radial symmetry;

\[
\frac{\partial f}{\partial t} = \frac{D}{\sinh^2 \rho} \frac{\partial}{\partial \rho} \left( \sinh^2 \rho \frac{\partial f}{\partial \rho} \right)
\]
The geodesic cylindrical coordinate

\[ v_1 = \frac{p_1}{E} = \tanh \eta \]
\[ v_2 = \frac{p_2}{E} = \frac{1}{\cosh \eta} \tanh \xi \cos \phi \]
\[ v_3 = \frac{p_3}{E} = \frac{1}{\cosh \eta} \tanh \xi \sin \phi \]

\[ E = m \cosh \eta \cosh \xi = m_T \cosh \eta \]
\[ p_1 = m \sinh \eta \cosh \xi = m_T \sinh \eta \]
\[ p_T = \sqrt{p_2^2 + p_3^2} = m \sinh \xi \]
\[ m_T = \sqrt{p_T^2 + m^2} \]

longitudinal rapidity \( \eta = \ln \frac{E + p_1}{m_T} \)
transverse rapidity \( \xi = \ln \frac{m_T + p_T}{m} \)

Identity \( E/m = \cosh \rho = \cosh \eta \cosh \xi \)

\( \rho = \xi \), when \( \eta = 0 \)
\( \rho = \ln \frac{E + |p|}{m} \)

We can analyse transvers momentum (rapidity) distributions at fixed longitudinal rapidity \( (\eta = 0, 2.2, \cdots) \), using the equation,

\[ f(\rho, t) = C \frac{\rho}{\sinh \rho} \exp \left[ -\frac{\rho^2}{2\sigma(t)^2} \right] \]

Parameters \( C, \quad \sigma(t)^2 = 2Dt \)
3. Analysis of $P_t$ distributions observed at the RHIC

<table>
<thead>
<tr>
<th>Centrality</th>
<th>$m$</th>
<th>$C$</th>
<th>$\sigma(t)^2$</th>
<th>$\chi^2_{min}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>00-05%</td>
<td>0.51</td>
<td>380.5 ± 11.5</td>
<td>0.356 ± 0.001</td>
<td>149.08</td>
</tr>
<tr>
<td>05-10%</td>
<td>0.50</td>
<td>316.0 ± 9.4</td>
<td>0.362 ± 0.001</td>
<td>119.96</td>
</tr>
<tr>
<td>10-20%</td>
<td>0.49</td>
<td>235.4 ± 7.1</td>
<td>0.372 ± 0.001</td>
<td>115.20</td>
</tr>
<tr>
<td>20-30%</td>
<td>0.48</td>
<td>166.6 ± 5.0</td>
<td>0.378 ± 0.001</td>
<td>97.94</td>
</tr>
<tr>
<td>30-40%</td>
<td>0.45</td>
<td>111.1 ± 3.3</td>
<td>0.396 ± 0.002</td>
<td>67.8</td>
</tr>
<tr>
<td>40-60%</td>
<td>0.43</td>
<td>59.0 ± 1.8</td>
<td>0.409 ± 0.002</td>
<td>67.8</td>
</tr>
<tr>
<td>60-80%</td>
<td>0.36</td>
<td>26.8 ± 0.8</td>
<td>0.451 ± 0.002</td>
<td>25.9</td>
</tr>
</tbody>
</table>

STAR($\eta = 0$) error: 10% \( n.d.f. = 35 - 3 \)

<table>
<thead>
<tr>
<th>Centrality</th>
<th>$m$</th>
<th>$C$</th>
<th>$\sigma(t)^2$</th>
<th>$\chi^2_{min}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>00-10%</td>
<td>0.71</td>
<td>121.0 ± 0.7</td>
<td>0.307 ± 0.001</td>
<td>351.2</td>
</tr>
<tr>
<td>10-20%</td>
<td>0.67</td>
<td>88.7 ± 0.6</td>
<td>0.325 ± 0.001</td>
<td>233.5</td>
</tr>
<tr>
<td>20-40%</td>
<td>0.63</td>
<td>56.9 ± 0.7</td>
<td>0.340 ± 0.001</td>
<td>97.8</td>
</tr>
<tr>
<td>40-60%</td>
<td>0.57</td>
<td>27.5 ± 0.5</td>
<td>0.358 ± 0.002</td>
<td>28.7</td>
</tr>
</tbody>
</table>

BRAHMS($\eta = 0$) : statistical error only \( n.d.f. = 26 - 3 \)

<table>
<thead>
<tr>
<th>Centrality</th>
<th>$m$</th>
<th>$C$</th>
<th>$\sigma(t)^2$</th>
<th>$\chi^2_{min}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>00-10%</td>
<td>0.80</td>
<td>52286 ± 1575</td>
<td>0.470 ± 0.002</td>
<td>93.7</td>
</tr>
<tr>
<td>10-20%</td>
<td>0.84</td>
<td>41669 ± 1561</td>
<td>0.452 ± 0.002</td>
<td>30.5</td>
</tr>
<tr>
<td>20-40%</td>
<td>0.81</td>
<td>19364 ± 862</td>
<td>0.469 ± 0.003</td>
<td>68.9</td>
</tr>
<tr>
<td>40-60%</td>
<td>0.70</td>
<td>6070 ± 418</td>
<td>0.524 ± 0.005</td>
<td>3.1</td>
</tr>
</tbody>
</table>

BRAHMS($\eta = 2.2$) : statistical error only \( n.d.f. = 11 - 3 \)
STAR (200GeV) $\eta=0$

$\left( \frac{n_e^+ + n_e^-}{2} \right)$

- $0-5\%$
- $10-20\%/10$
- $15/10$
- $30-40\%/20$
- $35/20$
- $60-80\%/30$
- $70/30$

$m=0.14$
BRAHNS (200GeV) $\eta=0.0$

$\frac{(p^+ + p^-)}{2}$

$\sigma^2(t) = 2Dt$

mass (GeV/c²)

$p_t$ (GeV/c) vs. centrality cut (%)
4. Summary

A formula proposed by Voloshin is derived from the stochastic model (Ornstein-Uhlenbeck process). However, it is gaussian in $p_t$, and observed $p_t$ distributions at the RHIC cannot be described by the formula.

A stochastic process in the three dimensional hyperbolic space $H^3$ is taken as a model of particle production processes. The solution is gaussian-like in radial rapidity.

$P_t$ distributions at $\eta = 0$ observed by the BRAHMS collaboration and those at $\eta = 0$ and $\eta = 2.2$ observed by the STAR collaboration are analysed.

Observed $p_t$ distributions are well described by the formula (gaussian in rapidity variable), if value of mass contained in rapidity is used as parameter.

Variance $\sigma^2(t)$ increases as the centrality cut increases. Mass $m$ decreases as the centrality cut increases.

Estimated values of $\sigma^2(t)$ and $m$ from the STAR collaboration are larger than those from the BRAHMS collaboration at the same centrality cut.