Skyrmions Coupled with Gravity

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Field Theories couples with Gravity

- Nonlinear field theories coupled to *gravity* lead to globally regular gravitating configurations
- Boundary Conditions: Black hole solutions with nonlinear hair arise
- Hairy black holes represent examples demonstrating that Israel's theorem does not generalize to theories with **non-Abelian fields**
- These black hole solutions are *asymptotically flat* and possess a regular event horizon
- Outside their horizon they retain the features of the corresponding **gravitating solitons**, and may thus be viewed as bound states of solitons and Schwarzschild black holes

Gravitating Solitons

- In the Einstein-Skyrme model the nonlinear chiral field theory describing baryons and nuclei in terms of Skyrmions is coupled to gravity
- The corresponding gravitating skyrmions and black holes with skyrmion hair exhibit a characteristic dependence on the **coupling parameter**.
- Static spherically and axially symmetric have been obtained numerically
  - *Gravitating skyrmions* in the $SU(2)$ and $SU(3)$ case
  - *Black holes with skyrmion hair* in the $SU(2)$ and $SU(3)$ case
- In flat space also skyrmions with no rotational but only **discrete symmetries** have been constructed
- Gravitating skyrmion configurations with only platonic symmetries; *tetrahedral and cubic symmetry*

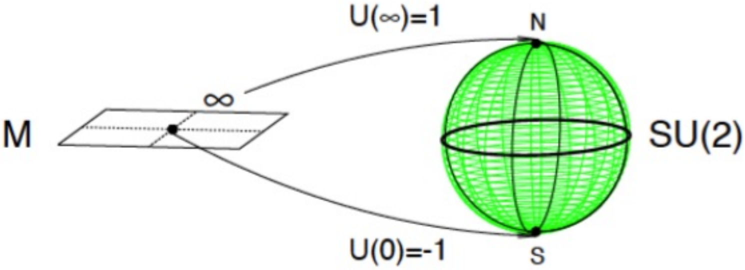
Einstein-Skyrme Model

- Action

$$S = \int \left[\frac{\mathcal{R}}{16\pi G} + \frac{\kappa^2}{4} \mathrm{tr}\left(K_\mu K^\mu\right) + \frac{1}{32e^2} \mathrm{tr}\left(\left[K_\mu, K_\nu\right]\left[K^\mu, K^\nu\right]\right)\right] \sqrt{-g}\, d^4x$$

$\mathcal{R}$: *Ricci scalar*; G: *Newton's constant*; g: determinant of the *metric*; $K_\mu = \partial_\mu U U^{-1}$; $U : M \to SU(2) \sim S^3$ and κ, e: *coupling constants*

- BC for finite-energy: $U \to 1$ at $|x^\mu| \to \infty$



One asymptotic region: ∞ treated as a point

Baryon Number

- Baryon Number due to the homotopy $\pi_3(SU(2)) = Z$ there is a topological charge:

$$B = \frac{1}{24\pi^2} \int \varepsilon_{ijk} \,\mathrm{tr}\left(K_i K_j K_k\right) \, d^3x$$

  where ε_{ijk} is the (constant) fully antisymmetric tensor ($e_{123} = 1$). The field configurations represent three dimensional gravitating topological solitons or antisolitons.

- *Dimensionless* radial coordinates $x = ker$
- *Dimensionless* coupling (gravitational) parameter $\alpha = 4\pi G k^2$

Einstein-Skyrme Equation

- Einstein equations: variation of action with respect to $g^{\mu\nu}$

$$G_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\mathcal{R} = 8\pi G T_{\mu\nu} \ ,$$

with stress-energy tensor $T_{\mu\nu} = g_{\mu\nu}\mathcal{L}_M - 2\frac{\partial \mathcal{L}_M}{\partial g^{\mu\nu}}$

$$T_{\mu\nu} = -\frac{\kappa^2}{2}\mathrm{tr}\left(K_\mu K_\nu - \frac{1}{2} g_{\mu\nu} K_\alpha K^\alpha\right) - \frac{1}{8e^2}\mathrm{tr}\left(g^{\alpha\beta}\left[K_\mu, K_\alpha\right]\left[K_\nu, K_\beta\right] - \frac{1}{4} g_{\mu\nu}\left[K_\alpha, K_\beta\right]\left[K^\alpha, K^\beta\right]\right)$$

- Field equations: variation with respect to the Skyrme field

$$\nabla_\mu \left(\kappa^2 K^\mu + \frac{1}{4e^2}[K_\nu, [K^\mu, K^\nu]]\right) = 0$$

- Metric for platonic symmetries

$$ds^2 = -f dt^2 + \frac{l}{f}\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right)$$

$f = f(r, \theta, \phi)$ and $l = l(r, \theta, \phi)$

- Skyrme Field

$$\boxed{U = e^{2ih(r)(P-1/2)}}$$

Here $h(r)$ is the profile function and $P(\xi, \bar{\xi})$ is a 2×2 hermitian projector where $\xi = e^{i\phi} \tan(\theta/2)$ angular variables

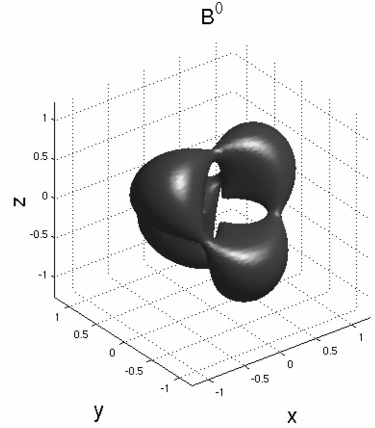
- Projector $P(\xi, \bar{\xi})$ a mapping from $S^2 \to CP^1$

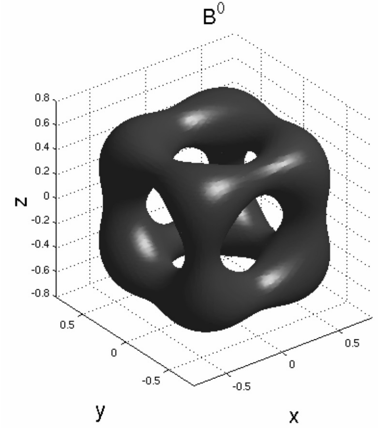
$$P(V) = \frac{V \otimes V^\dagger}{|V|^2}\ , \qquad V = (R(\xi), 1)^t$$

- *Tetrahedral* $B = 3$ and *cubic* $B = 4$ ansätz

$$\begin{aligned} R_{\rm tetra} &= \frac{\sqrt{3}\xi^2 + 1}{\xi(\xi^2 - \sqrt{3})} \\ R_{\rm cube} &= \frac{\xi^4 + 2\sqrt{3}i\xi^2 + 1}{\xi^4 - 2\sqrt{3}i\xi^2 + 1} \end{aligned}$$

Platonic Gravitating Skyrmions

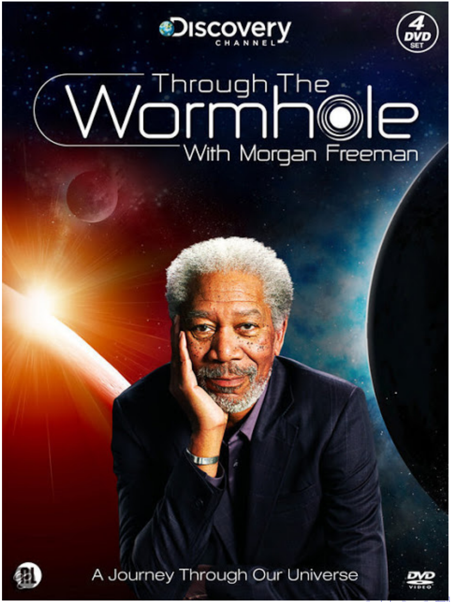




$B = 3$ baryon density

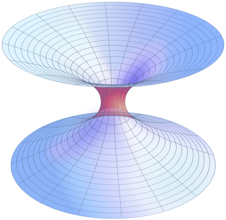
$B = 4$ baryon density

Science Fiction or Not

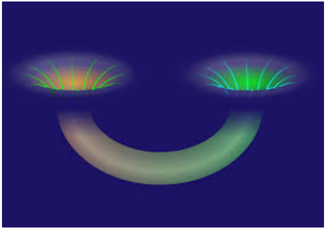


Wormhole

- Among the plethora of solutions of the Einstein equations: there are also the wormhole
- Wheeler: connect two *different universes* or *distant regions of same universe* (a **shortcut** through space-time that would collapse if one tried to pass them)



Inter-Universe Wormhole



Intra-Universe Wormhole

Motivation

- Traversable wormholes: be crossed in both directions; no trivial topology; no horizon
- For that, *a new kind of matter* was coupled to gravity and in which energy-momentum tensor would **violate all** (null, weak and strong) energy conditions *in order to keep the throat of the wormhole open*
- Good candidate: *phantom field*, i.e. a *scalar field* with a **reversed sign** in front of its kinetic term
- Phatom field in a variety of settings like *stars and neutron star*; and *Einstein-Gauss-Bonnet-dilaton theories*
- Our model: A **phantom field** and **chiral fields** minimally couple to Einstein gravity
- Motive: **the presence of non-Abelian fields can lead to new interesting gravitational phenomena, since "hairy black holes" were discovered in the Einstein-Skyrme model**

Wormhole Model

- Einstein action gravity coupled to a phantom field and ordinary matter fields

$$\boxed{S = \int \left[\frac{1}{16\pi G} \mathcal{R} + \mathcal{L}_{\rm ph} + \mathcal{L}_{\rm ch} + \mathcal{L}_{\rm sk} \right] \sqrt{-g}\; d^4x}$$

- Lagrangian of the phantom field ϕ:

$$\boxed{\mathcal{L}_{\rm ph} = +\frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi}$$

- Static spherically symmetric wormhole solutions so line element would be

$$ds^2 = -A^2 dt^2 + d\eta^2 + R^2 d\Omega^2 \ ,$$

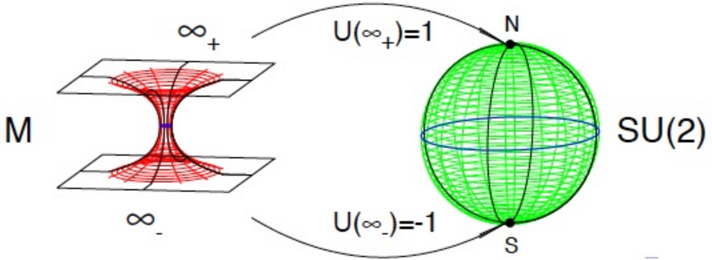
where $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$: the metric of the unit sphere, while $A = A(\eta)$ and $R = R(\eta)$ for $-\infty < \eta < \infty$

- Chiral Field

$$\boxed{U = \cos F + i \sin F \ \vec{e} \cdot \vec{\tau}}$$

with the unit vector field: $\vec{e} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$; $\vec{\tau}$: Pauli matrices; $F = F(\eta)$ is the chiral profile function; **baryon number one**

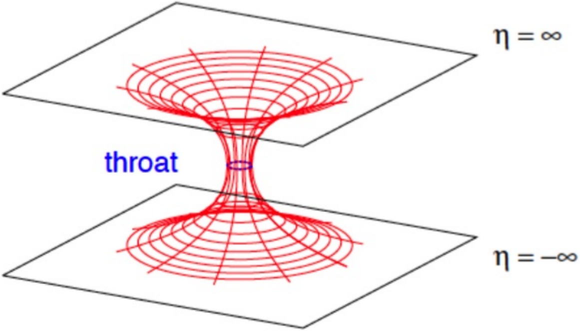
- The limits $\eta \to \pm\infty$: two disjoint asymptotically flat regions



Two asymptotic region: $\pm\infty$ treated as points

Wormhole Characteristics

- Three parameters characterise the solutions: the the gravitational parameter α; the radius r_0 of the throat (*surface of minimal area*) of the wormhole; the value of the chiral fields F_0 at the throat.
- Dependence: of the wormhole solutions on these parameters
- Geometry: of the wormhole changes with α.
- Stability: is an essential question for transversable wormholes



Boundary Conditions

- The first condition fixes the position of the throat; the second the areal radius ; and the third the value of the chiral profile function at the throat *a free parameter*

$$\boxed{\eta = 0, \quad R(0) = r_0 \ , \quad F(0) = F_0}$$

- Geometry of the throat: $\alpha < \alpha_{cr}$ and $\alpha > \alpha_{cr}$

$$\alpha_{cr} = \frac{1}{\sin^2 F_0 \left[2 + \frac{sin^2 F_0}{e^2 r_0^2}\right]}$$

- Asympotic Regions: The first condition sets the time scale, whereas the second and third result from requiring finite energy and conserved topological charge

$$\boxed{A(\eta \to \infty) \to 1 \ , \quad F(\eta \to \infty) \to 0 \ , \quad F(\eta \to -\infty) \to \pi}$$

Properties of the Wormholes

- The mass M of the solutions is obtained from the asymptotic behaviour of the metric function A,

$$\boxed{A \to 1 - \frac{\mu}{\eta}}$$

  where the dimensionless mass parameter μ is related to M by $\mu = \alpha M / M_0$, with $M_0 = 4\pi\kappa$

- The ODEs are solved numerically for the given set of boundary conditions in terms of the free parameters
  - α gravitational parameter
  - r_0 the throat of the wormhole

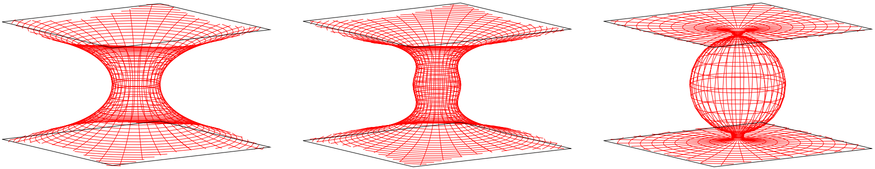
NLS wormholes: $1/e^2 \to 0$

- Geometry dependence only on α, Einstein-matter equations are **scale invariant so** $r_0 = 1$
- $\alpha \to 0$: the chiral fields have no influence upon the wormhole geometry **(single throat)**
- $\alpha > \alpha_{\rm cr} = 1/2$: the wormhole possesses a double throat with a belly in the interior
- $\alpha \to 1$: the chiral fields are *fully localized inside this inner region*, while the areal radius of the two throats decreases to zero.

  The space-time then splits into the following parts:

  The outer parts correspond to empty Minkowski spaces; the inner part corresponds to an Einstein universe with a chiral field carrying a topological charge one

NLS wormholes

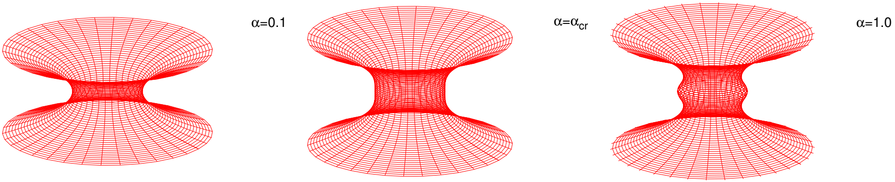


Three dimensional view of the isometric embedding for (a) $\alpha = \mathbf{0.001}$, (b) $\alpha = \mathbf{0.8}$ and (c) $\alpha = \mathbf{0.99999}$

Skyrmionic Wormholes

- Two free parameters the gravitational coupling α and the throat size r_0
- Domain of existence
    - $r_0 \to \infty$ very large and $\alpha < 1$ becomes the NLS wormhole
    - $\alpha > 1$ exist up to a maximal value $r_0 < r_{0,\mathrm{max}}$
- **As α increases an increasing influence of the matter on the geometry of the throat and the formation of two throats with an inner belly beyond a critical value**
- Boundary conditions asymmetrically with respect to the throat, we can distribute the chiral fields asymmetrically with respect to the two asymptotically flat universes
- Deformation of the wormhole by the matter: asymmetric
- Two throats develop asymmetrically with respect to their location and size

Skyrmionic Wormholes



Three dimensional view of the isometric embedding of the wormholes when $r_0 = 1$ while (a) $\alpha = \mathbf{0.1}$, (b) $\alpha = \alpha_{\rm cr}$ and (c) $\alpha = \mathbf{1.0}$.

Stability Analysis

- The stability of wormholes is crucial for their physical relevance.
- Eigenvalue problem: Linear stability analysis in the spherically symmetric sector
- $\omega^2 < 0$, the perturbation increase in time (unstability)
- NLS wormholes: unstable
- Skyrmionic Wormholes: $\omega^2 = \omega(r_0)$ for different values of α
- **Skyrmionic wormholes are, also, unstable**
- Nonsymmetric Skyrmionc Wormholes: unstable

Conclusions

- Skyrmionic wormholes with topological charge greater than one
- **Axially symmetric skyrmionic wormholes**
- BPS gravitating skyrmions and back holes
- BPS Skyrmionic wormholes

Relevant Papers

- *Physical Review D* 87 084069 (2013), gr-qc/1302.5560
- *Physics Letters* B 643 (2006) 213-220, gr-qc/0608110
- *Physics Letters* B 635 (2006) 161-167, gr-qc/0601103
- *Classical Quantum Gravity* 22 (2005) 1077-1086, hep-th/0406209
- *Physics Letters* B 600 (2004) 116-125, gr-qc/0407035
- *Physical Review* D 69 (2004) 124035, hep-th/0311161

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