Defect manifolds and Skyrmions

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June 19, 2016

Introduction

- Space-time can have large fluctuations of the metric and topology at scales of order of the Planck lenght (10^{-33} cm) (Wheeler).
- This leads to the picture of the space-time foam: space-time appears smooth and nearly flat at large-scales but is highly curved and and has non-trivial topology at small-scales.
- This non-trivialities at small-scales lead to interesting phenomena:
  - The propagation of particles is affected
  - Possible mass generation mechanism
  - ...
- *Defect manifolds* can be used as a model for describing the space-time foam. These manifolds are constructed from Minkowski space time by "surgery".

Defect manifolds

$$\begin{aligned} M^0 &= \mathbb{R}^3 \setminus \{\bar{x},\ |\bar{x}| < b\} \\ M^1 &= \mathbb{R}^3 \setminus \{\bar{x},\ |\bar{x}| < b \wedge \bar{x} \in \partial B_b, \bar{x} \cong P_1(\bar{x})\} \\ M^2 &= \mathbb{R}^3 \setminus \{\bar{x},\ |\bar{x}| < b \wedge \bar{x} \in \partial B_b, \bar{x} \cong P_2(\bar{x})\} \end{aligned}$$

where

$$\begin{aligned} P_1((x,y,z)) &= -(x,y,z) \\ P_2((x,y,z)) &= (x,-y,z). \end{aligned}$$

Therefore

$$\mathcal{M}_4^{\tau=0} = \mathbb{R}\times M^0 \simeq \mathbb{R}\times\left(\mathbb{S}^3 \setminus \{p\}\right) \quad , \quad \mathcal{M}_4^{\tau=1} = \mathbb{R}\times M^1 \simeq \mathbb{R}\times\left(\mathbb{R}P^3 \setminus \{p\}\right)$$

$$\text{and} \quad \mathcal{M}_4^{\tau=2} = \mathbb{R}\times M^2 \simeq \mathbb{R}\times\left(\frac{\mathbb{S}^3}{R} \setminus \{p\}\right)$$

Examples

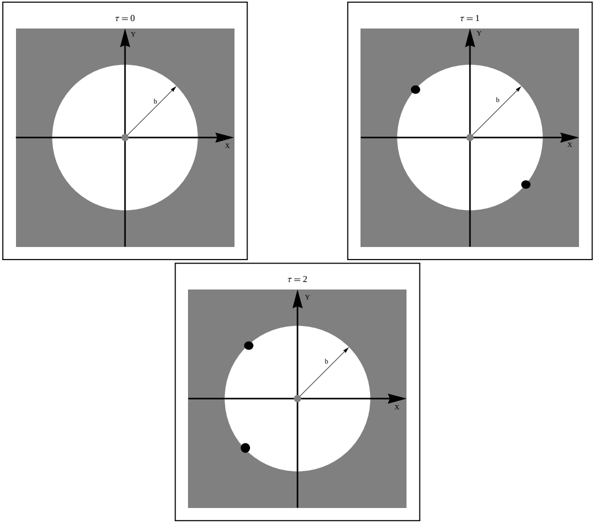

Figure: Two dimensional representation of the manifolds/orbifolds $\mathcal{M}_4^{(\tau)}$

It is easy to construct a manifold with more than one defect:

$M^{\tau}_{x_i,b}$: defect of size b at x_i

$$M^{\tau}_{\{x_1,\ldots,x_N\},b} = \bigcap_{i=1}^{N} M^{\tau}_{x_i,b}$$

and for a infinite distribution (space-time foam)

$$\mathcal{M} = \mathbb{R} \times \left(\lim_{N\to\infty} M^{\tau_i}_{\{x_1,\ldots,x_N\},b_i} \right)$$

Scalar example

Let us start with a massless scalar field in four dimensions obeying the Klein-Gordon equation

$$\Box\Phi^{(\tau)} = 0$$

where $\Phi^{(\tau)}$ is the scalar field defined in the manifold $\mathcal{M}^{\tau}$. The Green's function

$$\begin{aligned} G(x,x') &= -i\langle 0|T\Phi^{(\tau)}(x)\Phi^{(\tau)}(x')|0\rangle \\ \Box G(x,x') &= -\delta^{(3)}(\bar{x}-\bar{x}')\delta(t-t'). \end{aligned}$$

After Fourier transform $G_{\omega}(\bar{x},\bar{x}') = \int dt e^{-i\omega(t-t')}G(x,x')$:

$$\left(\omega^2+\nabla^2\right)G_{\omega}(\bar{x},\bar{x}') = \delta^{(3)}(\bar{x}-\bar{x}')$$

Let us assume that $\mathcal{M}^{\tau}$ has a static spherical defect at the origin then:

$$G_\omega(r,r',\gamma)_{r>r'} = \sum_{n=0}^{\infty} a_n \frac{1}{\sqrt{r}} H^{(1)}_{n+1/2}\left(|\omega| r\right) P_n(\cos\gamma)$$

$$G_\omega(r,r',\gamma)_{r<r'} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{r}} \left(b_n H^{(1)}_{n+1/2}\left(|\omega| r\right) + c_n H^{(2)}_{n+1/2}\left(|\omega| r\right)\right) P_n(\cos\gamma)$$

The coefficients a_n, b_n and c_n are fixed by the conditions

- 1. Boundary conditions at the defect surface.
- 2. Continiuty at $r = r'$.
- 3. Jump condition of the first derivative at $r = r'$.

The topology of the defect is encoded in the first condition.

- Let us consider $\mathcal{M}^{\tau=0}$.
- Assume that the defect size b is small.
- Impose Sommerfeld radiation condition at $r=\infty$.

$$G_\omega(r,r',\gamma) = G_{\omega,free}(\bar{x},\bar{x}') - \frac{be^{i|\omega|(r'+r)}}{4\pi rr'} + \mathcal{O}\left((|\omega|b)^2\right)$$

and

$$G_{\omega,free}(\bar{x},\bar{x}') = \lim_{b\to 0} G_\omega(\bar{x},\bar{x}') = \frac{e^{i|\omega||\bar{x}-\bar{x}'|}}{4\pi|\bar{x}-\bar{x}'|}$$

The energy momentum tensor for this scalar theory is

$$T_{\mu\nu} = \frac{2}{3}\phi_{,\mu}\phi_{,\nu} - \frac{1}{6}\eta_{\mu\nu}\eta^{\sigma\rho}\phi_{,\sigma}\phi_{,\rho} - \frac{1}{3}\phi\phi_{,\mu\nu} + \eta_{\mu\nu}\frac{1}{12}\phi\Box\phi$$

We can determine the VEV of the energy momentum tensor as follows

$$\begin{aligned}\langle T_{\mu\nu}\rangle &= i\lim_{x'^{\mu}\to x^{\mu}}(+\frac{2}{3}\partial_{\mu}\partial_{\nu'} - \frac{1}{6}\eta_{\mu\nu}\eta^{\sigma\rho}\partial_{,\sigma}\partial_{,\rho} - \frac{1}{3}\partial_{,\mu}\partial_{.\nu} \\ &+ \frac{1}{12}\partial_{\mu'}\partial^{\mu'})\left(G_{\omega}(x,x',\gamma) - G_{\omega,free}(x,x',\gamma)\right)\end{aligned}$$

$$\langle T_{\mu\nu}\rangle = -\frac{b^2}{32\pi^2 r^6}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{2r^2-3x^2}{3r^2} & -\frac{xy}{r^2} & -\frac{xz}{r^2} \\ 0 & -\frac{xy}{r^2} & \frac{2r^2-3y^2}{3r^2} & -\frac{zy}{r^2} \\ 0 & -\frac{xz}{r^2} & -\frac{zy}{r^2} & \frac{2r^2-3z^2}{3r^2} \end{pmatrix} + \mathcal{O}((b|\omega|)^3).$$

$\lambda\Phi^4$ *model*

$$\mathcal{L} = \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \lambda\Phi^4$$

1 loop Green's function

$$G(x,x') = G^{(0)}(x,x') + \lambda G^{(1)}(x,x')$$

with

$$G^{(1)}(x,x') = -\frac{i}{2}\int d^4z G^{(0)}(x,z)G^{(0)}(z,z)G^{(0)}(z,x').$$

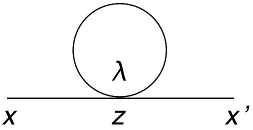

Figure: One loop correction

Therefore at 1 loop order

$$(\Box_x + m^2_{\text{eff}})\left(G^{(0)}(x,x') + \lambda G^{(1)}(x,x')\right) = -\delta^{(3)}(\bar{x}-\bar{x}')\delta(t-t')$$

where $m^2_{eff} \propto \lambda$. Solving for m^2_{eff}

$$m^2_{\text{eff}} = i\frac{\lambda}{2}G^{(0)}(x,x) = \frac{\lambda b}{16\pi^2}\frac{1}{r^3} + \mathcal{O}(b^2/r^2)$$

- The defect in the space time leads to the generation of mass!
- $\lim_{b\to 0}\mathcal{M}^{\tau=0} = \mathcal{M}_4 \Rightarrow \lim_{b\to 0} m^2_{eff} = 0$
- One may expect that in gas of defects the coordinate dependence of the generated mass to be replaced by some characteristic distance ($\mathbf{l}_{\text{foam}}$).

Gravitating defect Skyrmions

The main goal of this section is to answer the following question:

- Is it possible to have non-singular finite-energy solutions of the Einstein equations with non-trivial topology at small length scales?

$$S[g,\Omega] = \int_{\mathcal{M}_4^{\tau=1}} d^4X\sqrt{-g}\left[\frac{1}{16\pi G}R + \mathcal{L}_S\right]$$

$$\mathcal{L}_S = \frac{f^2}{4}g^{\mu\nu}\mathrm{tr}(\omega_\mu\omega_\nu) + \frac{1}{16e^2}g^{\mu\nu}g^{\rho\sigma}\mathrm{tr}([\omega_\mu,\omega_\rho][\omega_\nu,\omega_\sigma])$$

and

$$\mathcal{M}_4^{\tau=1} = \mathbb{R}\times M^1$$

$$\omega_\mu = \Omega^{-1}\partial_\mu\Omega, \quad \Omega\in SO(3)$$

Note that $M^1 \cup \{p\} \simeq \mathbb{R}P^3 \simeq SO(3)$, therefore

$$\Omega : \mathbb{R}P^3 \longrightarrow SO(3), \quad \deg \Omega = n$$

We use the charge 1 *ansatz*

$$\Omega = \cos \tilde{F}(r^2)\mathbb{I}_3 - \sin \tilde{F}(r^2)\hat{x} \cdot \vec{S} + \left(1 - \cos \tilde{F}(r^2)\right) \hat{x} \otimes \hat{x}$$

with boundary conditions

$$\tilde{F}(b^2) = \pi \quad \text{and} \quad \tilde{F}(\infty) = 0$$

where

$$S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad S_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Boundary conditions at the defect surface for the scalar field:

$$\Omega(r,\hat{x})|_{r=b} = -\mathbb{I}_3 + 2\hat{x}\otimes\hat{x} = \Omega(r,-\hat{x})|_{r=b}$$

A particular cover of $\mathcal{M}_4^{\tau=1}$ uses three charts. In one of this three charts the metric *ansatz* can be written as follows

$$\begin{aligned} ds^2 &= -\tilde{\mu}(W)^2 dT^2 + \left(1 - b^2/W\right)\tilde{\sigma}(W)^2 dY^2 + W\left(dZ^2 + \sin^2 Z dX^2\right) \\ W &\equiv b^2 + Y^2 \end{aligned}$$

$$X \in (0,\pi), \quad Y \in (-\infty,\infty), \quad Z \in (0,\pi)$$

Boundary conditions at the defect surface for the metric:

$$\tilde{\sigma}(\infty) = \pi, \quad \tilde{\mu} = 1$$

We are ready to solve the Einstein equations:

Solutions

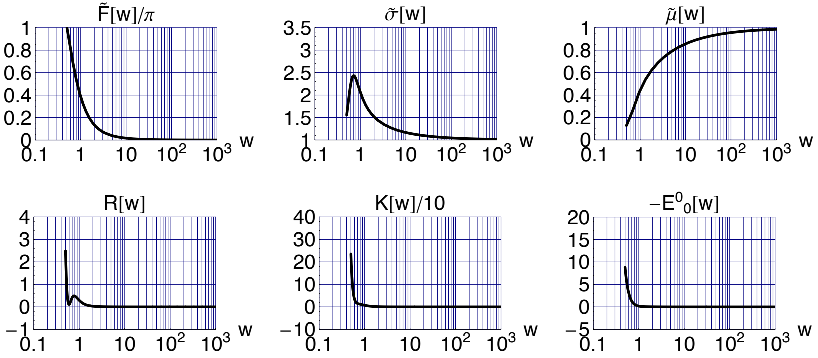

Figure: Solutions: R Ricci scalar, K Kreutschmann scalar, $E^0{}_0$ 00 component of the Einstein tensor

Phys.Rev. D90 (2014) no.2, 024007

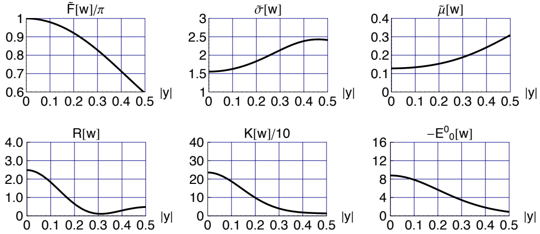

Figure: Behavior at the defect core

- The solutions are regular in M^1 (and also inside the defect core).
- The metric is asymptotically flat.
- E^0_0 behaves asymptotically as $1/Y^4$.
- It seems that for $y_{\rm crit} = e\, f\, b_{\rm crit} < 1/(2\sqrt{2})$ regular solutions do not exist (it remains to be confirmed)

Conclusions

- We have presented a framework for describing the space-time foam
- The motion of particles in the space-time foam (in our case a single defect space-time) can be very much affected
  - Modification of the propagators
  - The VEV of the EMT is modified ($\Rightarrow$ Casimir effect)
  - Mass generation
- We have succeeded in constructing a non-singular finite-energy solution of Einstein equations with non-trivial topology on small length scales.
- This Skyrmion solution combines the non-trivial topology of the space-time manifold and the field configuration manifold.
- Next steps:
  - critical size of the defect?
  - non-singular solution including BPS term?

$$+\frac{1}{2}\text{Thanks}$$