A long-standing problem: Isospin symmetry breaking in the BPS Skyrme model

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Solitons, (non) Integrability and Geometry V
Skyrmions - from atomic nuclei to neutron stars

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Introduction: The isospin symmetry breaking

- **Tony Hilton Royle Skyrme (1922-1987):** Producer

EFT: The Skyrme model (as one of the most popular and successful)

- Baryons as "vortices" in a mesonic fluid $\Rightarrow$ topological solitons, i.e., emergent, *non-perturbative* objects

$$\mathscr{L} = \lambda_0 \mathscr{L}_0 + \underbrace{\underbrace{\lambda_2 \mathscr{L}_2 + \lambda_4 \mathscr{L}_4}_{\text{massless } \mathscr{L}_{\text{Skyrme}}}}_{\text{massive } \mathscr{L}_{\text{Skyrme}}} + \lambda_6 \mathscr{L}_6$$

$$\mathscr{L}_2 = -\mathrm{Tr}(L_\mu L^\mu), \quad \mathscr{L}_4 = \mathrm{Tr}([L_\mu, L_\nu]^2), \quad \mathscr{L}_6 = -\pi^4 \mathcal{B}_\mu \mathcal{B}^\mu$$

$$\mathcal{B}^\mu = \frac{1}{24\pi^2} \mathrm{Tr}(\epsilon^{\mu\nu\rho\sigma} L_\nu L_\rho L_\sigma), \quad L_\mu = U^\dagger \partial_\mu U, \quad U \in \mathrm{SU}(2) \cong \mathbb{S}^3$$

- Baryon number $\equiv$ topological charge: $B = \int d^3x \mathcal{B}^0$

The BPS Skyrme model:

$$\mathcal{L}_{06} = -\frac{\lambda^2}{24^2}\mathrm{Tr}(\epsilon^{\mu\nu\rho\sigma}U^\dagger\partial_\mu UU^\dagger\partial_\nu UU^\dagger\partial_\rho UU^\dagger\partial_\sigma U)^2 - \mu^2\mathcal{U}(\mathrm{Tr}U)$$

- Parametrization of the Skyrme field

$$U = e^{i\xi\vec{n}\cdot\vec{\sigma}} = \cos\xi + i\sin\xi\vec{n}\cdot\vec{\sigma} \qquad \vec{n}^2 = 1$$

$$\vec{n} = \frac{1}{1+|u|^2}\left(u+\bar{u}, -i(u-\bar{u}), 1-|u|^2\right)$$

$$\mathcal{L}_{06} = \frac{\lambda^2\sin^4\xi}{(1+|u|^2)^4}\left(\epsilon^{\mu\nu\rho\sigma}\xi_\nu u_\rho\bar{u}_\sigma\right)^2 - \mu^2\mathcal{U}(\xi)$$

C. Adam, J. Sanchez-Guillen, A. Wereszczynski, Phys. Lett. B **691**, 105 (2010).

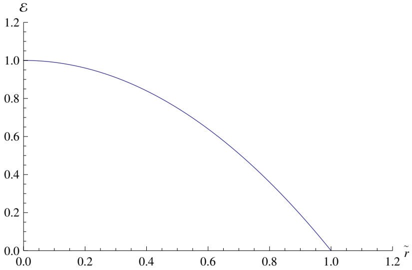
Exact solution: the standard Skyrme potential

$$\mathcal{U} = \mathcal{U}_\pi = \frac{1}{2}\mathrm{Tr}(1 - U) \;\; \rightarrow \;\; \mathcal{U}(\xi) = 1 - \cos\xi$$

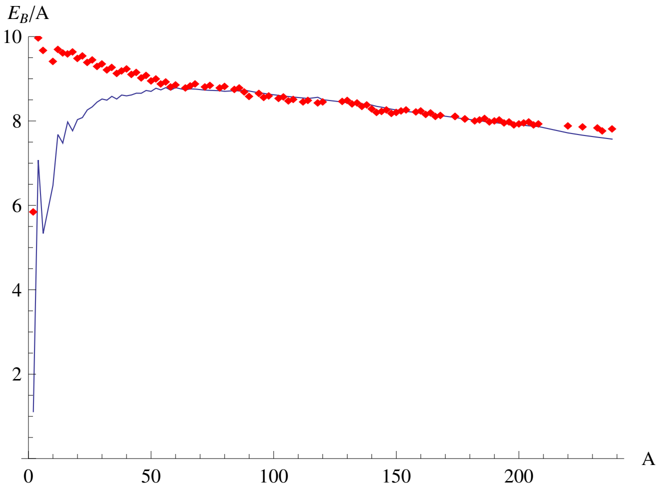
With the ansatz: $\xi = \xi(r), \qquad u(\theta, \phi) = \tan\frac{\theta}{2} e^{iB\phi}$

- Compact solution with radius proportional to $B^{1/3}$.
- Energy linear in the baryon number:

$$E = \frac{64\sqrt{2}\pi}{15}\mu\lambda|B|$$



- **Nuclear binding energies in the BPS Skyrme model**



C. Adam, C. Naya, J. Sanchez-Guillen, and A. Wereszczynski, Phys. Rev. Lett. **111**, 232501 (2013)

Contributions to the energy:

- Collective coordinate quantization of spin $\vec{J}$ and isospin $\vec{I}$ (j and $i_3 = (1/2)(Z - N)$ known)

$$U(t, \vec{x}) = \mathrm{A}(t) U_0(R_\mathrm{B}(t)\vec{x}) \mathrm{A}^\dagger(t)$$

$\Rightarrow$ Two copies of a symmetric top in the general case.

- Coulomb energies (most important for large B)

$$E_\mathrm{C} = \frac{1}{2\varepsilon_0} \int d^3x d^3x' \frac{\rho(\vec{r})\rho(\vec{r}\,')}{4\pi|\vec{r} - \vec{r}\,'|}$$

- Explicit isospin breaking: $M_\mathrm{n} > M_\mathrm{p}$

$$\Rightarrow E = E_\mathrm{sol} + E_\mathrm{rot} + E_\mathrm{C} + E_\mathrm{I}$$

Isospin breaking:

$$E_{\mathrm{I}} = a_{\mathrm{I}} i_3 \quad \text{where} \quad a_{\mathrm{I}} < 0 \quad \Leftrightarrow \quad M_{\mathrm{n}} > M_{\mathrm{p}}$$

- Binding energy of nucleus $X = {}^{A}_{Z}X$, $N = A - Z$:

$$E_{\mathrm{B},X} = ZE_{\mathrm{p}} + NE_{\mathrm{n}} - E_X$$

- Three free parameters $\lambda, \mu, a_{\mathrm{I}}$: fit to 3 nuclear masses

$$\begin{aligned} M_{\mathrm{p}} &= 938.272\ \mathrm{MeV} \\ M_{\mathrm{n}} - M_{\mathrm{p}} &= 1.29333\ \mathrm{MeV} \\ M({}^{138}_{56}\mathrm{Ba}) &= 137.905\ \mathrm{u} \quad \text{where} \quad \mathrm{u} = 931.494\ \mathrm{MeV} \end{aligned}$$

Take 1: The isospin-breaking potential

Isospin-breaking potential: Perturbative contribution to the Lagrangian

$$\mathcal{U}_{\mathrm{I}} = \frac{\eta^2}{4}\mathrm{Tr}(1 - \tau_3 U \tau_3 U^\dagger)$$

Iso-rotations, $U \to AUA^\dagger$, are no longer symmetries.

- Small fluctuations of the Skyrme field $U = e^{i\vec{\pi}\cdot\vec{\tau}}$

$$\mu^2\mathcal{U} = \frac{\mu^2}{2}\mathrm{Tr}(1 - U) \approx \frac{\mu^2}{2}\vec{\pi}^2$$

$$\mathcal{U}_{\mathrm{I}} = \frac{\eta^2}{4}\mathrm{Tr}(1 - \tau_3 U \tau_3 U^\dagger) \approx \frac{\eta^2}{2}(\pi_1^2 + \pi_2^2 - \pi_3^2)$$

- Mass term ($\mathscr{L}_0$ contribution):

$$\Rightarrow \mu^2\mathcal{U} + \mathcal{U}_{\mathrm{I}} \approx \frac{1}{2}(\mu^2 + \eta^2)(\pi_1^2 + \pi_2^2) + \frac{1}{2}(\mu^2 - \eta^2)\pi_3^2 \quad ✔$$

Semiclassical quantization

$$U(\vec{r}, t) = A(t)U_0(R(B(t))\vec{r})A^\dagger(t) = AU_RA^\dagger$$

- Plugging into the $\mathcal{U}_\mathrm{I}$ potential

$$\mathcal{U}_I = \frac{\eta^2}{4}\mathrm{Tr}(1 - \tau_3 AU_0A^\dagger\tau_3 AU_0^\dagger A^\dagger) = \eta^2 \sin\xi[1 - (\vec{R}\cdot\vec{n})^2]$$

with $R_i = R_{3i}(A) = \frac{1}{2}\mathrm{Tr}[\tau_3 A\tau_i A^\dagger]$, i.e. $(A = a_0 + ia_i\tau_i)$,

$$R_1 = 2(a_0a_2+a_1a_3), \qquad R_2 = 2(a_2a_3-a_0a_1), \qquad R_3 = a_0^2-a_1^2-a_2^2+a_3^2$$

- Total contribution to the energy

$$\mathrm{U_I} = \frac{4\pi}{3}\frac{32}{35}\frac{|B|\lambda}{\sqrt{2}\mu}\eta^2\langle N|3-R_1^2-R_2^2-R_3^2|N\rangle = \frac{8\pi}{3}\frac{32}{35}\frac{|B|\lambda}{\sqrt{2}\mu} \neq \mathcal{U}_I(i_3) \quad ✘$$

Take 2: The covariant derivative approach

New terms: Perturbative inclusion of Dirichlet + Isospin-breaking

$$\mathcal{L}_2 - \mathcal{U}_\mathrm{I} = -\alpha \mathrm{Tr}(L_\mu L^\mu) - \frac{\eta^2}{4}\mathrm{Tr}(1 - \tau_3 U \tau_3 U^\dagger)$$

- For static solutions U_0

$$\mathcal{L}_2 - \mathcal{U}_\mathrm{I} = -\alpha \mathrm{Tr}(U_0^\dagger D_\mu U_0 U_0^\dagger D^\mu U_0)$$

- Covariant derivative

$$D_\mu U_0 = \partial_\mu U_0 + i\frac{\omega}{2}[\tau_3, U_0]\delta_{\mu 0}$$

with ω defined as

$$\omega^2 = -\frac{\eta^2}{2\alpha}$$

- Equivalent to introduce an iso-rotated field

$$\tilde{U} = T U_0 T^\dagger \in \mathrm{SU}(2)$$

by the time-dependent SU(2) matrix

$$T = \begin{pmatrix} e^{i\frac{\omega t}{2}} & 0 \\ 0 & e^{-i\frac{\omega t}{2}} \end{pmatrix} = \cos\frac{\omega t}{2} + i\tau_3 \sin\frac{\omega t}{2}$$

E. Rathske, Z. Phys. A **331**, 499 (1988)

$$\Rightarrow U_0^\dagger D_\mu U_0 = \tilde{U}^\dagger \partial_\mu \tilde{U}$$

- Re-write the perturbative contribution

$$\tilde{\mathscr{L}}_2 = \mathscr{L}_2 - \mathcal{U}_{\mathrm{I}} = -\alpha \mathrm{Tr}(\tilde{U}^\dagger \partial_\mu \tilde{U} \tilde{U}^\dagger \partial^\mu \tilde{U})$$

makes life easy (quantization)!

Semiclassical quantization

- Usual time-dependent rotations and iso-rotations

$$U_0(\vec{r}) \rightarrow U(\vec{r},t) = A(t)U_0(R(B(t))\vec{r})A^\dagger(t) = AU_RA^\dagger$$

- Iso-rotations and Euler angles

$$A = a_0 + i\vec{a}\cdot\vec{\tau} = V_3(\alpha)V_2(\beta)V_3(\gamma)$$

- Usual parametrization of the BPS part, $\mathscr{L}_{06}$.
- Usual parametrization of the perturbative part, $\tilde{\mathscr{L}}_2$, but with a modified A matrix

$$\tilde{A} = TA = V_3(\alpha+\omega t)V_2(\beta)V_3(\gamma)$$

  - Shifted isospin velocities

$$\tilde{A}^\dagger\dot{\tilde{A}} = i(\vec{\mathcal{W}}+\vec{D})\cdot\frac{\vec{\tau}}{2}$$

with $\quad D = \omega(R_1, R_1, R_3), \qquad R_i = R_{3i}(A) = \frac{1}{2}\mathrm{Tr}[\tau_3 A\tau_i A^\dagger]$

- Perturbative contribution

$$
\begin{aligned}
\tilde{\mathscr{L}}_2 = \quad & \mathcal{W}_i\Big[-\alpha\mathrm{Tr}(T_iT_j)\Big]\mathcal{W}_j + 2\mathcal{W}_i\Big[-\alpha\,\epsilon_{jkl}\,x_k\mathrm{Tr}(T_iL_l)\Big]\omega_j \\
& +\omega_i\Big[-\alpha\,\epsilon_{ikl}\,\epsilon_{jmn}\,x_k\,x_m\mathrm{Tr}(L_lL_n)\Big]\omega_j + D_i\Big[-\alpha\mathrm{Tr}(T_iT_j)\Big]D_j \\
& +2\mathcal{W}_i\Big[-\alpha\mathrm{Tr}(T_iT_j)\Big]D_j + 2D_i\Big[-\alpha\,\epsilon_{jkl}\,x_k\mathrm{Tr}(T_iL_l)\Big]\omega_j - \mathcal{E}_2.
\end{aligned}
$$

- BPS contribution

$$
\begin{aligned}
\mathscr{L}_{60} = \quad & \mathcal{W}_i\Big[\frac{9\lambda^2}{24^2}\mathrm{Tr}(\epsilon^{pqr}T_iL_qL_r)\mathrm{Tr}(\epsilon^{pst}T_jL_sL_t)\Big]\mathcal{W}_j \\
& +2\mathcal{W}_i\Big[\frac{18\lambda^2}{24^2}\epsilon_{jkl}\,x_k\mathrm{Tr}(\epsilon^{pqr}T_iL_qL_r)\mathrm{Tr}(\epsilon^{pst}L_lL_sL_t)\Big]\omega_j \\
& +\omega_i\Big[\frac{9\lambda^2}{24^2}\epsilon_{ikl}\,\epsilon_{jmn}\,x_k\,x_m\mathrm{Tr}(\epsilon^{pqr}L_lL_qL_r)\mathrm{Tr}(\epsilon^{pst}L_nL_sL_t)\Big]\omega_j - \mathcal{E}_0,
\end{aligned}
$$

Total Lagrangian

$$L = \frac{1}{2}\mathcal{W}_i\mathcal{I}_{ij}\mathcal{W}_j - \mathcal{W}_i\mathcal{K}_{ij}\omega_{ij} + \frac{1}{2}\omega_i\mathcal{J}_{ij}\omega_j + \mathcal{W}_i A_i + \omega_i B_i - \mathrm{U_I} - E,$$

- Additional contributions

$$A_i = -2\alpha D_j \int d^3x \mathrm{Tr}(T_i T_j)$$

$$B_i = -2\alpha D_j \epsilon_{ikl} \int d^3x x_k \mathrm{Tr}(T_j L_l)$$

$$V_I = \alpha D_i D_j \int d^3x \mathrm{Tr}(T_i T_j)$$

$$E = E_0 + E_2$$

- Moments of inertia tensors

$$\mathcal{I}_{ij} = \int d^3x \left[-2\alpha \mathrm{Tr}(T_i T_j) + \frac{18\lambda^2}{24^2} \mathrm{Tr}(\epsilon^{pqr} T_i L_q L_r) \mathrm{Tr}(\epsilon^{pst} T_j L_s L_t) \right]$$

$$\mathcal{J}_{ij} = \epsilon_{ikl}\, \epsilon_{jmn}\, x_k\, x_m \int \left[-2\alpha \mathrm{Tr}(L_l L_n) + \frac{18\lambda^2}{24^2} \mathrm{Tr}(\epsilon^{pqr} L_l L_q L_r) \mathrm{Tr}(\epsilon^{pst} L_n L_s L_t) \right]$$

$$\mathcal{K}_{ij} = \epsilon_{jkl}\, x_k \int d^3x \left[2\alpha\, \mathrm{Tr}(T_i L_l) - \frac{18\lambda^2}{24^2} \mathrm{Tr}(\epsilon^{pqr} T_i L_q L_r) \mathrm{Tr}(\epsilon^{pst} L_l L_s L_t) \right]$$

- Canonical momenta

$$K_i = \frac{\partial L}{\partial \mathcal{W}_i} = \mathcal{I}_{ij}\mathcal{W}_j - \mathcal{K}_{ij}\omega_j + A_i \qquad L_i = \frac{\partial L}{\partial \omega_i} = -\mathcal{W}_j\mathcal{K}_{ji} + \mathcal{J}_{ij}\omega_j + B_i$$

- Hamiltonian

$$H = \quad K_i\mathcal{W}_i + L_i\omega_i - L = K_i\mathcal{W}_i + L_i\omega_i - \frac{1}{2}\mathcal{W}_i\mathcal{I}_{ij}\mathcal{W}_j + \mathcal{W}_i\mathcal{K}_{ij}\omega_j$$
$$-\frac{1}{2}\omega_i\mathcal{J}_{ij}\omega_j - \mathcal{W}_iA_i - \omega_iB_i + \mathrm{U_I} + E$$

which after substituting the angular velocities reads

$$H = \quad \frac{1}{2}\sum_{i=1}^{3}\left[\frac{(L_i + \mathcal{K}_i\frac{K_i}{\mathcal{I}_i})^2}{\mathcal{J}_i - \frac{\mathcal{K}_i^2}{\mathcal{I}_i}} + \frac{K_i^2}{\mathcal{I}_i} + \frac{(B_i + \mathcal{K}_i\frac{A_i}{\mathcal{I}_i})^2}{\mathcal{J}_i - \frac{\mathcal{K}_i^2}{\mathcal{I}_i}} + \frac{A_i^2}{\mathcal{I}_i} - \frac{1}{\mathcal{I}_i - \frac{\mathcal{K}_i^2}{\mathcal{J}_i}}K_iA_i\right.$$
$$\left. - \frac{\mathcal{K}_i}{\mathcal{I}_i\mathcal{J}_i - \mathcal{K}_i^2}K_iB_i - \frac{\mathcal{K}_i}{\mathcal{I}_i\mathcal{J}_i - \mathcal{K}_i^2}L_iA_i - \frac{1}{\mathcal{J}_i - \frac{\mathcal{K}_i^2}{\mathcal{I}_i}}L_iB_i\right] + \mathrm{U_I} + E$$

Axially symmetric ansatz:

$$U = \cos\xi + i\sin\xi\,\vec{n}\cdot\vec{\tau}, \qquad \xi = \xi(r), \quad \vec{n} = (\sin\theta\cos B\phi, \sin\theta\sin B\phi, \cos\theta)$$

- Additional contributions

$$A_i = \frac{32}{3}\pi\alpha\omega R_i \int dr r^2 \sin^2\xi$$

$$B_1 = B_2 = 0, \qquad B_3 = -\frac{32\pi}{3}\alpha\omega B R_3 \int d^3x r^2 \sin^2\xi$$

$B = 1$

$$B_i = -\frac{32\pi}{3}\alpha\omega R_i \int dr r^2 \sin^2\xi$$

- Moments of inertia tensors

$$\mathcal{I}_1 = \mathcal{I}_2 = \int \left[\frac{32\pi}{3} \alpha \sin^2 \xi + \frac{1 + 3n^2}{3r^2} \pi \lambda^2 \xi_r^2 \sin^4 \xi \right] r^2 dr$$

$$\mathcal{I}_3 = \int \left[\frac{32\pi}{3} \alpha \sin^2 \xi + \frac{4}{3r^2} \pi \lambda^2 \xi_r^2 \sin^4 \xi \right] r^2 dr$$

$$\mathcal{J}_1 = \mathcal{J}_2 = \int \left[\frac{8}{3} (3 + B^2) \pi \alpha \sin^2 \xi + \frac{4}{3r^2} \pi \lambda^2 B^2 \xi_r^2 \sin^4 \xi \right] r^2 dr$$

$$\mathcal{J}_3 = B\mathcal{K}_3 = B^2 \mathcal{I}_3, \qquad \mathcal{K}_1 = \mathcal{K}_2 = 0$$

$B = 1$

$$\mathcal{I}_1 = \mathcal{I}_2 = \mathcal{I}_3 = \int \left[\frac{32\pi}{3} \alpha \sin^2 \xi + \frac{4}{3r^2} \pi \lambda^2 \xi_r^2 \sin^4 \xi \right] r^2 dr$$

$$\mathcal{J}_1 = \mathcal{J}_2 = \mathcal{J}_3 = \mathcal{K}_1 = \mathcal{K}_2 = \mathcal{K}_3 = \mathcal{I}_1$$

- Canonical momenta

$$\vec{K} = (\mathcal{I}_1\mathcal{W}_1 + A_1, \mathcal{I}_1\mathcal{W}_2 + A_2, \mathcal{I}_3(\mathcal{W}_3 - B\omega_3) + A_3)$$

$$\vec{L} = (\mathcal{J}_1\omega_1, \mathcal{J}_1\omega_2, -B\mathcal{I}_3(\mathcal{W}_3 - B\omega_3) - BA_3)$$

- Hamiltonian

$$H = E + \mathrm{U_I} + \frac{1}{2}\left[\frac{\vec{L}^2}{\mathcal{J}_1} + \frac{1}{\mathcal{I}_1}(\vec{K} - \vec{A})^2 + \left(\frac{1}{\mathcal{I}_3} - \frac{1}{\mathcal{I}_1}\right)(K_3 - A_3)^2 - \frac{B^2}{\mathcal{J}_1}K_3^2\right]$$

$$(\vec{K} - \vec{A})^2 = \vec{K}^2 + \vec{A}^2 - 2\vec{K}\cdot\vec{A} = \vec{K}^2 + A^2 + 2AI_3$$

where $\quad A_i = AR_i, \quad A = \frac{32}{3}\pi\alpha\omega\int dr r^2 \sin^2\xi$

Problem!

Covariant derivative $\equiv$ Isospin chemical potential

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Skyrmion semiclassical quantization in the presence of an isospin chemical potential

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The semiclassical description of Skyrmions at small isospin chemical potential μ_I is carefully analyzed. We show that when the calculation of the energy of a nucleon is performed using the straightforward generalization of the vacuum sector techniques ($\mu_I = 0$), together with the "natural" assumption $\mu_I = \mathcal{O}(N_c^0)$, the proton and neutron masses are nonlinear in μ_I in the regime $|\mu_I| < m_\pi$. Although these nonlinearities turn out to be numerically quite small, such a result fails to strictly agree with the very robust prediction that for those values of μ_I the energy excitations above the vacuum are linear in μ_I. The resolution of this paradox is achieved by studying the realization of the large N_c limit of QCD in the Skyrme model at finite μ_I. This is done in a simplified context devoid of the technical complications present in the Skyrme model but which fully displays the general scaling behavior with N_c. The analysis shows that the paradoxical result appears as a symptom of using the semiclassical approach beyond its regime of validity and that, at a formal level, the standard methods for dealing with the Skyrme model are only strictly justified for states of high isospin, $I \sim N_c$.

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Take 3: An isospin-broken solution

New potential for the BPS Skyrme model:

$$\mathcal{U} = \frac{1}{2}\mathrm{Tr}(1-U) + \epsilon^2 \frac{1}{2} \frac{\mathrm{Tr}(1-\tau_3 U \tau_3 U^\dagger)}{\mathrm{Tr}(1+U)}$$

$$= 2\sin^2\left(\frac{\xi}{2}\right)\left(1+\epsilon(1-n_3^2)\right) \neq \mathcal{U}(r)$$

- BPS equation

$$\frac{\lambda^2}{r^2}\sin^4\xi\xi_r^2\left(\frac{i\epsilon_{ij}\nabla_i u \nabla_j \bar{u}}{(1+|u|^2)^2}\right)^2 = \mu^2\mathcal{U}$$

- Assuming $u = v(\theta)e^{i\phi}$

$$\alpha^2\frac{\lambda^2}{r^4}\sin^4\xi\xi_r^2 = 2\mu^2\sin^2\left(\frac{\xi}{2}\right)$$

$$\frac{1}{\alpha^2}\left(\frac{2v\,v_\theta}{\sin\theta(1+v^2)^2}\right)^2 = 1+\epsilon^2(1-n_3^2)$$

Exact static solution for $B = 1$

- Profile function

$$\xi = \begin{cases} 2\arccos\left(\frac{r}{R}\right) & r \in [0, R] \\ 0 & r \geq R \end{cases} \qquad R = \left(\frac{4\sqrt{2}\alpha\lambda}{\mu}\right)^{1/3}$$

- Angular part

$$\frac{1}{1 + v^2} = \frac{1}{2\epsilon}\left(1 + \sqrt{2}\sin\left[\frac{1}{2}\left(\beta - \frac{\pi}{2} + \beta\cos\theta\right)\right]\right)$$

with

$$\beta = \arcsin\left(\frac{2\epsilon - 1}{\sqrt{2}}\right) + \frac{\pi}{4}, \qquad \alpha = \frac{\beta}{4\epsilon}$$

- Limit $\epsilon \to 0 \Rightarrow \alpha = 1/2$

$$\xi = \arccos\left[\left(\frac{\mu}{2\sqrt{2}\lambda}\right)^{1/3} r\right], \qquad v = \tan\frac{\theta}{2}$$

Semiclassical quantization: Please, proceed as always

- Moments of inertia tensors

$$\mathcal{I}_{ij} = \frac{\lambda^2}{2} \int \xi_r^2 \sin^4 \xi \partial_l n_i \partial_l n_j d^3 x$$

$$\mathcal{J}_{ij} = \frac{\lambda^2}{4} \frac{\beta^2}{\epsilon^2} \int \frac{1}{r^4} \xi_r^2 \sin^4 \xi \, \cos^2 \theta (r^2 \delta_{ij} - x_i x_j) r^2 \sin \theta dr d\theta d\phi$$

$$\mathcal{K}_{ij} = -\frac{\lambda^2}{2} \int \xi_r^2 \sin^4 \xi \epsilon_{jkl} \epsilon_{def} x_k \partial_s n_i n_d \partial_s n_e \partial_l n_f d^3 x$$

- Radial integration

$$\int \xi_r^2 \sin^4 \xi dr = \frac{64 \cdot 2^{5/6}}{35} \left(\frac{\epsilon\mu}{\beta\lambda} \right)^{1/3}$$

Symmetries

$$\mathcal{I}_{11} = \mathcal{I}_{22} \neq \mathcal{I}_{33}, \qquad \mathcal{J}_{11} = \mathcal{J}_{22} \neq \mathcal{J}_{33}, \qquad \mathcal{K}_{11} = \mathcal{K}_{22} \neq \mathcal{K}_{33}$$

$$\mathcal{I}_{11} \neq \mathcal{J}_{11} \neq \mathcal{K}_{11}, \qquad \mathcal{I}_{33} = \mathcal{J}_{33} = \mathcal{K}_{33}$$

Examples for different ϵ

- $\epsilon = 0.1$

$$\mathcal{I}_{ij} = \lambda^2 \left(\frac{\mu}{\lambda}\right)^{1/3} \begin{pmatrix} 11.122 & 0 & 0 \\ 0 & 11.122 & 0 \\ 0 & 0 & 11.1604 \end{pmatrix}$$

$$\mathcal{J}_{ij} = \lambda^2 \left(\frac{\mu}{\lambda}\right)^{1/3} \begin{pmatrix} 11.1546 & 0 & 0 \\ 0 & 11.1546 & 0 \\ 0 & 0 & 11.1604 \end{pmatrix}$$

$$\mathcal{K}_{ij} = \lambda^2 \left(\frac{\mu}{\lambda}\right)^{1/3} \begin{pmatrix} 11.1301 & 0 & 0 \\ 0 & 11.1301 & 0 \\ 0 & 0 & 11.1604 \end{pmatrix}$$

Examples for different ϵ

- $\epsilon = 0.5$

$$\mathcal{I}_{ij} = \lambda^2 \left(\frac{\mu}{\lambda}\right)^{1/3} \begin{pmatrix} 11.5889 & 0 & 0 \\ 0 & 11.5889 & 0 \\ 0 & 0 & 12.0541 \end{pmatrix}$$

$$\mathcal{J}_{ij} = \lambda^2 \left(\frac{\mu}{\lambda}\right)^{1/3} \begin{pmatrix} 11.7519 & 0 & 0 \\ 0 & 11.7519 & 0 \\ 0 & 0 & 12.0541 \end{pmatrix}$$

$$\mathcal{K}_{ij} = \lambda^2 \left(\frac{\mu}{\lambda}\right)^{1/3} \begin{pmatrix} 11.6282 & 0 & 0 \\ 0 & 11.6282 & 0 \\ 0 & 0 & 12.0541 \end{pmatrix}$$

- Canonical momenta

$$K_i = \frac{\partial L}{\partial \mathcal{W}_i} \Rightarrow \vec{K} = (\mathcal{I}_1 \mathcal{W}_1 - \mathcal{K}_1 \omega_1, \mathcal{I}_1 \mathcal{W}_2 - \mathcal{K}_1 \omega_2, \mathcal{I}_3(\mathcal{W}_3 - \omega_3))$$

$$L_i = \frac{\partial L}{\partial \omega_i} \Rightarrow \vec{L} = (\mathcal{J}_1 \omega_1 - \mathcal{K}_1 \mathcal{W}_1, \mathcal{J}_1 \omega_2 - \mathcal{K}_1 \mathcal{W}_2, \mathcal{I}_3(\omega_3 - \mathcal{W}_3))$$

Remember! Body-fixed operators!

$$I_i = -R_{ij}(A) K_j, \qquad J_i = -R_{ij}(B)^T L_j \qquad \left(R_{ij}(A) = \frac{1}{2} \mathrm{Tr}[\tau_i A \tau_j A^\dagger] \right)$$

- Hamiltonian

$$\mathscr{H}_{\mathrm{rot}} = \frac{1}{2} \left\{ \frac{1}{\mathcal{J}_1 - \frac{\mathcal{K}_1^2}{\mathcal{I}_1}} \left[(L_1 + \frac{\mathcal{K}_1}{\mathcal{I}_1} K_1)^2 + (L_2 + \frac{\mathcal{K}_1}{\mathcal{I}_1} K_2)^2 \right] + \frac{K_1^2}{\mathcal{I}_1} + \frac{K_2^2}{\mathcal{I}_1} + \frac{K_3^2}{\mathcal{I}_3} \right\}$$

- Hamiltonian 1

$$\mathscr{H}_{\mathrm{rot}} = \frac{1}{2}\left[\frac{\vec{L}^2}{\mathcal{J}_1 - \frac{\mathcal{K}_1^2}{\mathcal{I}_1}} + \frac{\vec{K}^2}{\mathcal{I}_1 - \frac{\mathcal{K}_1}{\mathcal{J}_1}} + \left(\frac{1}{\mathcal{I}_3} - \frac{\mathcal{I}_1 + \mathcal{J}_1 - 2\mathcal{K}_1}{\mathcal{I}_1\mathcal{J}_1 - \mathcal{K}_1^2}\right) K_3^2\right] + \frac{\mathcal{K}_1}{\mathcal{I}_1\mathcal{J}_1 - \mathcal{K}_1^2}\vec{L}\cdot\vec{K}$$

- Hamiltonian 2

$$\mathscr{H}_{\mathrm{rot}} = \frac{1}{2}\left[\frac{\mathcal{K}_1}{\mathcal{I}_1\mathcal{J}_1 - \mathcal{K}_1^2}\vec{M}^2 + \frac{\mathcal{I}_1 - \mathcal{K}_1}{\mathcal{I}_1\mathcal{J}_1 - \mathcal{K}_1^2}\vec{L}^2 + \frac{\mathcal{J}_1 - \mathcal{K}_1}{\mathcal{I}_1\mathcal{J}_1 - \mathcal{K}_1^2}\vec{K}^2 + \left(\frac{1}{\mathcal{I}_3} - \frac{\mathcal{I}_1 + \mathcal{J}_1 - 2\mathcal{K}_1}{\mathcal{I}_1\mathcal{J}_1 - \mathcal{K}_1^2}\right) K_3^2\right]$$

Grand spin

$$\vec{M} = \vec{L} + \vec{K} \Rightarrow \vec{M}^2 = \vec{L}^2 + \vec{K}^2 + 2\vec{L}\cdot\vec{K}$$

Hamiltonian 1: Calculation of the scalar product $\vec{L} \cdot \vec{K}$

$$\vec{K} \cdot \vec{L} = \frac{1}{2} K_+ L_- + \frac{1}{2} K_- L_+ + K_3 L_3$$

- Ladder operators

$$K_+ = K_1 + i\,K_2, \qquad K_- = K_1 - i\,K_2, \qquad L_+ = L_1 + i\,L_2, \qquad L_- = L_1 - i\,L_2$$

- Action on an arbitrary state

$$K_\pm |i, i_3, k_3\rangle \otimes |j, j_3, l_3\rangle = \sqrt{i(i+1) - k_3(k_3 \pm 1)}\; |i, i_3, k_3 \pm 1\rangle \otimes |j, j_3, l_3\rangle$$

$$K_3 |i, i_3, k_3\rangle \otimes |j, j_3, l_3 \otimes\rangle = k_3 |i, i_3, k_3\rangle \otimes |j, j_3, l_3\rangle$$

$$L_\pm |i, i_3, k_3\rangle \otimes |j, j_3, l_3\rangle = \sqrt{j(j+1) - l_3(l_3 \pm 1)}\; |i, i_3, k_3\rangle \otimes |j, j_3, l_3 \pm 1\rangle$$

$$L_3 |i, i_3, k_3\rangle \otimes |j, j_3, l_3\rangle = l_3 |i, i_3, k_3\rangle \otimes |j, j_3, l_3\rangle$$

General nucleon state

$$|N\rangle = |1/2, i_3, k_3\rangle \otimes |1/2, j_3, l_3\rangle$$

- Non-vanishing actions

$$K_+ L_- |1/2, i_3, -1/2\rangle \otimes |1/2, j_3, 1/2\rangle = |1/2, i_3, 1/2\rangle \otimes |1/2, j_3, -1/2\rangle$$

$$K_- L_+ |1/2, i_3, 1/2\rangle \otimes |1/2, j_3, -1/2\rangle = |1/2, i_3, -1/2\rangle \otimes |1/2, j_3, 1/2\rangle$$

$$K_3 L_3 |1/2, i_3, k_3\rangle \otimes |1/2, j_3, l_3\rangle = k_3 l_3 |1/2, i_3, k_3\rangle \otimes |1/2, j_3, l_3\rangle$$

Constraint $k_3 = -l_3$ is always fulfilled!

- Matrix element

$$\langle N|\vec{K} \cdot \vec{L}|N\rangle = \langle N|K_3 L_3|N\rangle = k_3 l_3 = -\frac{1}{4}$$

Impossible to distinguish proton from neutron!

General nucleon state

$$|N\rangle = \frac{1}{\sqrt{2}}\left(|1/2, i_3, 1/2\rangle \otimes |1/2, j_3, -1/2\rangle + e^{i\alpha}|1/2, i_3, -1/2\rangle \otimes |1/2, j_3, 1/2\rangle\right)$$

- Different actions

$$K_+L_-|N\rangle = \frac{e^{i\alpha}}{\sqrt{2}}|1/2, i_3, 1/2\rangle \otimes |1/2, j_3, -1/2\rangle$$

$$K_-L_+|N\rangle = \frac{1}{\sqrt{2}}|1/2, i_3, -1/2\rangle \otimes |1/2, j_3, 1/2\rangle$$

$$K_3L_3|N\rangle = -\frac{1}{4}|N\rangle$$

- Matrix element

$$\langle N|\vec{K}\cdot\vec{L}|N\rangle = \frac{e^{i\alpha}}{4} + \frac{e^{-i\alpha}}{4} - \frac{1}{4} = \frac{1}{2}\cos\alpha - \frac{1}{4}$$

$$\text{if} \left.\begin{array}{l} \alpha = \pi \rightarrow \text{proton} \\ \alpha = 0 \rightarrow \text{neutron} \end{array}\right\} \Rightarrow \langle N|\vec{K}\cdot\vec{L}|N\rangle = -i_3 - \frac{1}{4}$$

Hamiltonian 2: Composition of grand spin $\vec{M}$

$$\vec{M} = \vec{K} + \vec{L} \Rightarrow m = 0, 1$$

- Corresponding states

$$|m, m_3\rangle \sim |i, k_3\rangle \otimes |j, l_3\rangle$$

- Two different states fulfilling $m_3 = k_3 + l_3 = 0$

$$|0,0\rangle = \frac{1}{\sqrt{2}} \left(|1/2, 1/2\rangle \otimes |1/2, -1/2\rangle - |1/2, -1/2\rangle \otimes |1/2, 1/2\rangle \right)$$

$$|1,0\rangle = \frac{1}{\sqrt{2}} \left(|1/2, 1/2\rangle \otimes |1/2, -1/2\rangle + |1/2, -1/2\rangle \otimes |1/2, 1/2\rangle \right)$$

$$\text{if} \quad \left. \begin{array}{l} m = 0 \rightarrow \text{proton} \\ m = 1 \rightarrow \text{neutron} \end{array} \right\} \Rightarrow M_n > M_p$$

Meeting both approaches: Equivalent states $\Rightarrow$ same energies

- Hamiltonian 1

$$H_{\mathrm{rot}} = \frac{1}{2}\left[\frac{\mathcal{I}_1}{\mathcal{I}_1\mathcal{J}_1 - \mathcal{K}_1^2}\, l(l+1) + \frac{\mathcal{J}_1}{\mathcal{I}_1\mathcal{J}_2 - \mathcal{K}_1^2}\, k(k+1) + \left(\frac{1}{\mathcal{I}_3} - \frac{\mathcal{I}_1 + \mathcal{J}_1 - 2\mathcal{K}_1}{\mathcal{I}_1\mathcal{J}_1 - \mathcal{K}_1^2}\right) k_3^2\right] + \frac{\mathcal{K}_1}{\mathcal{I}_1\mathcal{J}_1 - \mathcal{K}_1^2}\langle N|\vec{K}\cdot\vec{L}|N\rangle$$

- Hamiltonian 2

$$H_{\mathrm{rot}} = \frac{1}{2}\left[\frac{\mathcal{K}_1}{\mathcal{I}_1\mathcal{J}_1 - \mathcal{K}_1^2}\, m(m+1) + \frac{\mathcal{I}_1 - \mathcal{K}_1}{\mathcal{I}_1\mathcal{J}_1 - \mathcal{K}_1^2}\, l(l+1) + \frac{\mathcal{J}_1 - \mathcal{K}_1}{\mathcal{I}_1\mathcal{J}_1 - \mathcal{K}_1^2}\, k(k+1) + \left(\frac{1}{\mathcal{I}_3} - \frac{\mathcal{I}_1 + \mathcal{J}_1 - 2\mathcal{K}_1}{\mathcal{I}_1\mathcal{J}_1 - \mathcal{K}_1^2}\right) k_3^2\right]$$

- Equivalence relation

$$m(m+1) - j(j+1) - i(i+1) = 2\langle N|\vec{K}\cdot\vec{L}|N\rangle = -2i_3 - \frac{1}{2}$$

- For $i = j = 1/2$

$$\alpha = \arccos[m(m+1) - 1]$$

$$m = 0 \Rightarrow \alpha = \arccos(-1) = \pi \rightarrow \text{proton?}$$

$$m = 1 \Rightarrow \alpha = \arccos(1) = 0 \rightarrow \text{neutron?}$$

Analysis:

- Four different states corresponding to $\alpha = \pi$ ($m = 0$) and other four states for $\alpha = 0$ ($m = 1$); possible combinations of i_3 and j_3.

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- Conclusion: the Hamiltonian does not distinguish third components of spin and isospin $\Rightarrow$ Four ground states and four excited states.
- Effect of the isospin-breaking potential? To break the spherically symmetric solitons to axial ones (related to the body-fixed operators).

Conclusions

- Breaking of the isospin symmetry: Distinction between proton and neutron, binding energies.
- Long story: More than 4 years and three different attempts.
- Perturbative isospin-breaking potential:
  - Constant contribution to the energy.
- Covariant derivative approach:
  - Inclusion of isospin chemical potential (in-medium Skyrmions).
  - Not well-defined formalism.
- Isospin-broken solution:
  - Exact $B = 1$ solution for an isospin-breaking potential.
  - Axially symmetric soliton instead of spherical one.
  - No distinction between third components of isospin.

Please tell me and let us write the script