Hairy black holes in general Skyrme models

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based on C. Adam, O. Kichakova, Ya. Shnir, A. Wereszczynski, "Hairy black holes in the general Skyrme model", arXiv:1605.07625

June 21, 2016

Contents

Skyrme models

- Non-linear scalar field theories supporting top. solitons ("Skyrmions")
- Candidate low-energy EFT for QCD; (scalar) Skyrme field $\sim$ mesons
- Baryons and nuclei realized as top. solitons ("vortices" in "meson fluid")
- simplest case (two flavors): Skyrme field space = SU(2) (isospin) matrix U (three pions)
- top. degree of Skyrmion ("winding number " of map $\mathbb{R}^3_0 \simeq \mathbb{S}^3 \to \mathrm{SU}(2) \simeq \mathbb{S}^3$) = baryon number B
- Syms. of (two-flavor) QCD: (chiral) $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ broken to $\mathrm{SU}(2)_{\mathrm{iso}}$

Original Skyrme model

$$\mathcal{L} = \underbrace{\mathcal{L}_2 + \mathcal{L}_4}_{\mathcal{L}_{skyrme}} + \mathcal{L}_0 \,, \qquad \mathcal{L}_0 = -\mu^2 \mathcal{U}(\mathrm{Tr}(1 - U)), \quad \mathcal{U}(0) = 0$$

$$\text{e.g. } \mathcal{U}_\pi = \mathrm{Tr}(1 - U), \quad \mu = m_\pi \qquad \text{pot}$$

$$\mathcal{L}_2 = a\, g^{\mu\nu}\, \mathrm{Tr}\,(L_\mu L_\nu), \quad \mathcal{L}_4 = b\, \mathrm{Tr}\,([L_\mu, L_\nu]^2), \quad L_\mu = U^\dagger \partial_\mu U$$

($- + + +$ metric convention)

- Description of nucleons: 30% level precision
- Description of nuclei:
  Some successes: (iso-) spin excitational spectra
- Main problems:
  - too large binding energies: $\exists$ topological energy bound $E \geq cB$, but not saturated (non-BPS theory)
    But may be generalized to (near) BPS theory
  - Large B: crystals (not liquid)

Generalizations

- Poincare invariance & standard Hamiltonian (quadratic in time derivatives): quite restrictive

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_0 + \mathcal{L}_6$$

$$\mathcal{L}_6 = c\,|g|^{-1} g_{\mu\nu} \mathcal{B}^\mu \mathcal{B}^\nu , \quad \mathcal{B}^\mu = \frac{1}{24\pi^2} \mathrm{Tr}\,(\epsilon^{\mu\lambda\rho\sigma} L_\lambda L_\rho L_\sigma)$$

$\mathcal{B}_\mu \ldots$ baryon current with baryon number $B = \int d^3 x \mathcal{B}^0$

- Submodel $\mathcal{L}_0 + \mathcal{L}_6$ has BPS property
  also: perfect fluid EM-tensor; SDiff symmetries
- $\Rightarrow$ Possibility of generalized near-BPS Skyrme models

Skyrmions and Gravity

- Promote $g_{\mu\nu}$ to dyn. metric by adding EH action

$$S = \int d^4x |g|^{\frac{1}{2}} \left(\frac{1}{16\pi G} R + \mathcal{L}_{\mathrm{Sk}} \right)$$

- In principle: solve to describe self-gravitating Skyrmions, neutrons stars and hairy black holes
- In practise: needs simplification
- Static, radially symmetric metric (in Schwarzschild coord.)

$$ds^2 = -\sigma^2(r)N(r)dt^2 + \frac{dr^2}{N(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$N(r) = 1 - 2m(r)/r$.

- Symmetry reduction for matter:
  - Either macroscopic: fluid/solid $\epsilon(r)$, $p(r)$ with EoS $p = p(\epsilon)$
  - insert into Einstein eqs.
  - Or: sym. red. of field theory
  - Skyrme model: $U = \cos f + i \sin f \, \vec{n} \cdot \vec{\tau}$
  - Hedgehog ($B = 1$): $f = f(r)$,

$$\vec{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

  - Either calc. $T_{\mu\nu}$, insert into Einstein eqs. $\Rightarrow$ 3 ODEs
  - Or sym. red. of action $\mathcal{L}_{\rm Sk} \to -\mathcal{E}_{\rm Sk}(f, f', N)$

$$\Rightarrow S = -\int dt\, E, \quad E = \int d^3x \sqrt{|g|} \left(-\frac{1}{16\pi G} R + \mathcal{E}_{\rm Sk} \right)$$

$d^3x\sqrt{|g|} = d\Omega dr r^2 \sigma$

simpler derivation of field eqs.

$$E[f, m, \sigma] = \int dr\sigma \left(-\frac{m'}{G} + \mathcal{E}_r \right) + \text{b.t.}, \quad \mathcal{E}_r \equiv 4\pi r^2 \mathcal{E}_{\text{Sk}}$$

- varying σ (Lag. mult. ... constraint)

$$m' = G\,\mathcal{E}_r$$

- varying m

$$\frac{\sigma'}{\sigma} = \frac{2G}{r}\frac{\partial \mathcal{E}_r}{\partial N}$$

- varying f

$$\left(\frac{\partial}{\partial f} - \partial_r \frac{\partial}{\partial f'} \right) \sigma\,\mathcal{E}_r = 0$$

- Explicitly ($\alpha^2 = 4\pi G$; pion mass potential)

$$m' = \frac{\alpha^2}{2}\left[a\left(\frac{1}{2}r^2 N f'^2 + \sin^2 f\right) + b\sin^2 f\left(N f'^2 + \frac{\sin^2 f}{2r^2}\right) + \frac{cN}{2r^2} f'^2 \sin^4 f + m_\pi^2 r^2 \sin^2 \frac{f}{2}\right]$$

$$\frac{\sigma'}{\sigma} = \frac{1}{2}\alpha^2 f'^2\left(ar + \frac{2b\sin^2 f}{r} + \frac{c\sin^4 f}{r^3}\right)$$

$$f'' = \frac{1}{a + \frac{2b\sin^2 f}{r^2} + \frac{c\sin^4 f}{r^4}}\left\{-a\left[\left(\frac{2}{r} + \frac{N'}{N} + \frac{\sigma'}{\sigma}\right) f_r - \frac{\sin(2f)}{r^2 N}\right]\right.$$

$$-b\left(-\frac{2\cos f\sin^3 f}{r^4 N} + \frac{2f_r \sin^2 f}{r^2}\frac{N'}{N} + \frac{f_r{}^2\sin(2f)}{r^2} + \frac{2f_r\sin^2 f}{r^2}\frac{\sigma'}{\sigma}\right)$$

$$\left.+m_\pi^2\frac{\sin f}{2N} - c\left(-\frac{2f_r\sin^4 f}{r^5} + \frac{f_r\sin^4 f}{r^4}\frac{N'}{N} + \frac{2f_r{}^2\cos f\sin^3 f}{r^4} + \frac{f_r\sin^4 f}{r^4}\frac{\sigma'}{\sigma}\right)\right\}$$

- Generically, 4 integration constants
- $\sigma' \geq 0$ (see later)

- Regular solutions
  - $B = 1$: Self-gravitating Skyrmions
  - Boundary conditions

$$f(0) = \pi; \quad f(\infty) = 0; \quad m(0) = 0; \quad \sigma(\infty) = 1$$

$\Rightarrow$ discrete # of solutions (0,1,2)

  - $B >> 1$, spher.sym., e.g., neutron stars.
    - BPS submodel: longitude $\phi \to B\phi$
    - general Skyrme: NO minimizers ($f(0) = B\pi \ldots$ unstable)
    - macroscop. (fluid/crystal) description: $\epsilon(r)$ & $p_i(r)$ into E-eqs.

- Hairy black holes
  - $B = 1$: Boundary conditions ($r_h \ldots$ horizon radius)

$$N(r_h) = 0: \; m(r_h) = \frac{r_h}{2}; \quad f(\infty) = 0; \quad \sigma(\infty) = 1$$

  - Only 3 integration constants at $r = r_h$:
    $(Nf'')(r = r_h) = N(r_h) f''(r_h) = 0 \quad \Rightarrow \quad f'(r_h) = F(f_h, r_h)$
    $\Rightarrow$ discrete # of solutions (0,1,2) (for fixed r_h)
  - Here: free constants $f(r_h) \equiv f_h$, $\sigma(r_h) \equiv \sigma_h$ determined by b.c. at $r = \infty$
  - $B >> 1$: hairy BHs of nuclear matter?
    - BPS submodel: $\phi \to B\phi$ (but no hairy BHs)
    - general Skyrme: NO minimizers ($f(0) = B\pi \ldots$ unstable)
    - macroscop. (fluid/crystal) description: $\epsilon(r)$ & $p_i(r)$??

Black Holes with Skyrmion hair

- Known results
  - Standard Skyrme model $\mathcal{L}_2 + \mathcal{L}_4$ has both regular self-grav. solitons and hairy BHs
    H. Luckock and I. Moss, Phys. Lett. B **176** (1986) 341
    S. Droz, M. Heusler and N. Straumann, PLB **268** (1991) 371
    P. Bizon, T. Chmaj, Phys. Lett. B **297** (1992) 55
  - Recent result: BPS model $\mathcal{L}_6 + \mathcal{L}_0$ for *specific* potential $\mathcal{U} = 2f - \sin 2f$:
    NO hairy BH (despite regular solitons with/without gravity)
    S.B. Gudnason, M. Nitta, N. Sawado, JHEP 1512 (2015) 013
  - ?What happens in general? One motivation
  - see also: S.B. Gudnason, M. Nitta, N. Sawado, arXiv:1605.07954

BPS model

- adapt GNS1 for general one-vacuum potentials
  $\mathcal{U}(f=0)=0; \mathcal{U}_{,f}\geq 0,\ f\in[0,\pi]$
- Static, spher. sym. $\mathcal{L}_6+\mathcal{L}_0 = c|g|^{-1}g_{00}\mathcal{B}^0\mathcal{B}^0-\mu^2\mathcal{U}$
  $\mathcal{B}^0=-(1/2\pi^2)\sin\theta\sin^2 f\, f'$

$$\Rightarrow \mathcal{E}_r = \frac{c}{\pi^3}\frac{N}{r^2}\sin^4 f\, f'^2 + 4\pi\mu^2 r^2\mathcal{U}$$

$$\begin{aligned} 0 &= \left(\frac{\partial}{\partial f}-\partial_r\frac{\partial}{\partial f'}\right)\sigma\mathcal{E}_r \\ &= -\sin^2 f\,\partial_r\left(\frac{2c}{\pi^3}\frac{N\sigma}{r^2}\sin^2 f\, f'\right)+4\pi\mu^2 r^2\mathcal{U}_f \\ \Rightarrow&\quad \partial_r\left(\frac{N\sigma}{r^2}\sin^2 f\ f'\right)=2\pi^4\frac{\mu^2}{c}r^2\frac{\sigma\mathcal{U}_f}{\sin^2 f}\end{aligned}$$

- If $|f'(\infty)| < \infty$ (compacton: $|f'(R)| < \infty$), integrate

$$0 = \frac{N\sigma}{r^2} \sin^2 f \, f' \Big|_{r_h}^{R} = 2\pi^4 \frac{\mu^2}{c} \int_{r_h}^{R} dr \, r^2 \frac{\sigma \mathcal{U}_f}{\sin^2 f} > 0 \qquad !$$

- Compacton: if $|f'(R)| = \infty$ but $|\mathcal{B}^0(R)| \sim |f'(R) \sin^2 f(R)| < \infty$ ... proof slightly more complicated
- $\Rightarrow$ The BPS Skyrme model, for one-vacuum potentials, does NOT support hairy BHs (but supports stable top. solitons with/without gravity) ... First known case

Skyrme model $\mathcal{L}_2 + \mathcal{L}_4$

- Regular solutions: from Bizon, Chmaj (1992)
  - free parameter $f'(0)$. Shooting s.t. $f(\infty) = 0$
  - two branches (lower, stable and higher, unstable)
  - bifurcate at α_{max}

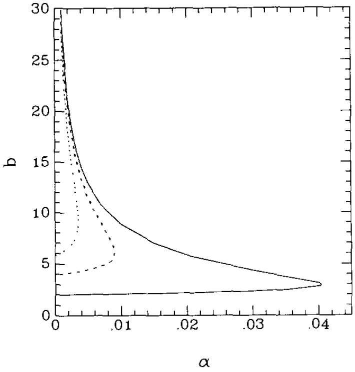


Fig. 1. Two fundamental branches of soliton solutions. The shooting parameter $b = -F'(0)$ as a function of α for $B = 1$ (solid line), $B = 2$ (dashed line), and $B = 3$ (dotted line).

- Hairy BH solutions: from Bizon, Chmaj (1992)
  - free parameter $f(r_h)$. Shooting s.t. $f(\infty) = 0$
  - two branches ("lower", stable and "higher", unstable)
  - bifurcate at $r_h^{\max}$
  - approach corresponding reg. sol. for $r_h \to 0$

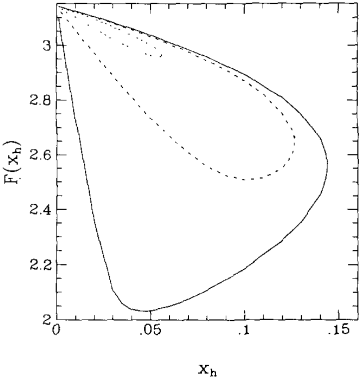
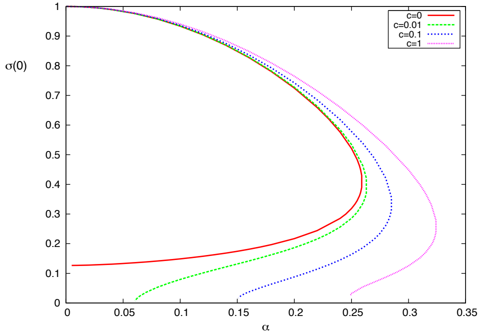


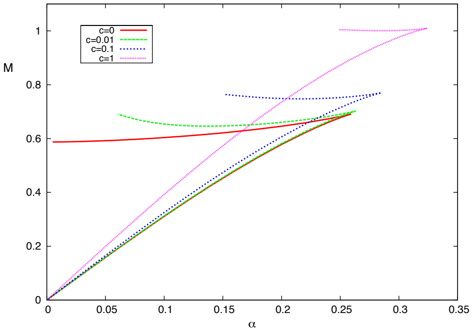
Fig. 2. Two fundamental branches of black hole solutions. The shooting parameter $F_H = F(x_H)$ as a function of x_H for $\alpha = 0.0005$ (solid line), $\alpha = 0.01$ (dashed line), and $\alpha = 0.03$ (dotted line).

General Skyrme model $\mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 - m_\pi^2 \mathcal{U}_\pi$

- Regular solutions:
  - $a = b = m_\pi = 1$. $\sigma(0)$ vs. α for some values of c
  - higher $\alpha_{\max}$ for higher c (repulsion)
  - two branches ("lower", stable and "higher", unstable)
  - lower approaches corresponding $G = 0$ sol.
  - higher branch does NOT go back to $\alpha = 0$ for $c \neq 0$



- Regular solutions:
  - $a = b = m_\pi = 1$. Rescaled ADM mass $m(\infty) = G\,M_{\rm ADM}$ vs. α for some values of c
  - lower mass (stable) and higher mass (unstable) branch
  - bifurcation at $\alpha_{\rm max}(c)$.



Hairy Black Holes

- Hairy BH solutions expected, at least for small r_h
- Discrete # of solutions for fixed r_h (two free constants f_h, σ_h, and two conditions $f(\infty) = 0$ and $\sigma(\infty) = 1$)
- Useful BH quantities: BH entropy $S = A_h/4 = \pi r_h^2$
- Black Hole surface gravity κ where

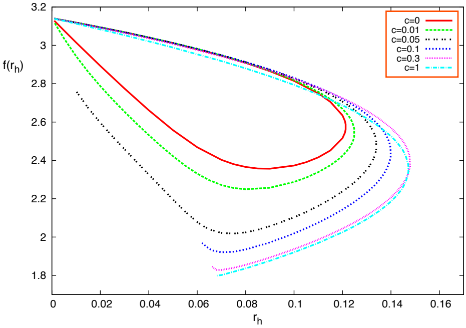
$$\kappa^2 = -\frac{1}{4} g^{tt} g^{rr} (\partial_r g_{tt})^2 \Big|_{r=r_h}$$

- Black Hole (Hawking) temperature T

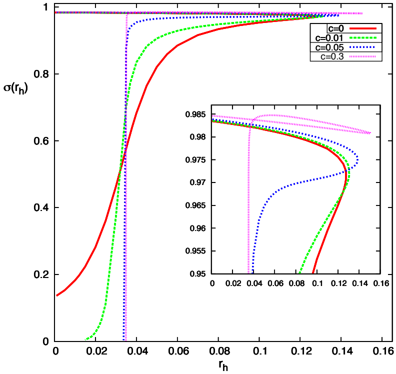
$$T = \frac{\kappa}{2\pi} = \frac{1}{4\pi} \sigma(r_h) N'(r_h)$$

- $\sigma(r_h) = 0 \;\Rightarrow\; T = 0 \;\ldots$ extremal BH

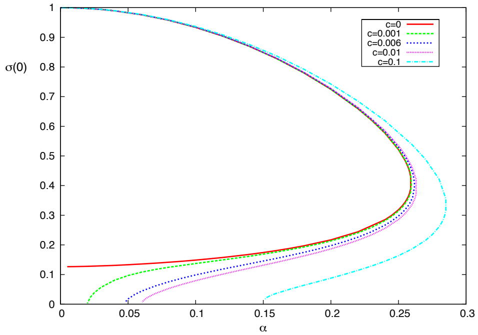
- Results:
  - f_h vs. r_h. $\alpha = 0.05$, $a = b = m_\pi = 1$ for some c
  - Upper (unstable) branch, $c > 0$:
    - f_h further decreases, does not approach $f_h = \pi$.
    - NO solution below certain $r_h^{\rm ex}(c)$



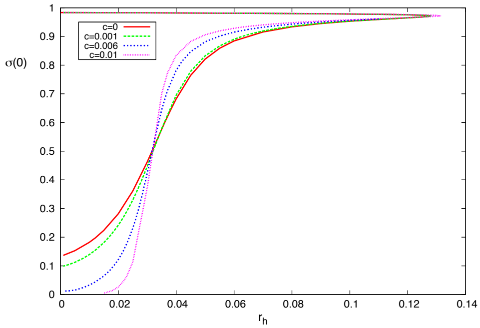
- Results:
  - σ_h vs. r_h. $\alpha = 0.05$, $a = b = m_\pi = 1$ for some c
  - Upper (unstable) branch, $c > 0$: $\sigma_h = 0$ for some $r_h^{\rm ex}(c)$
  - NO solution for $r_h < r_h^{\rm ex}(c)$
  - $r_h \searrow r_h^{\rm ex}$ ... extremal limit: extremal ($T = 0$) BH



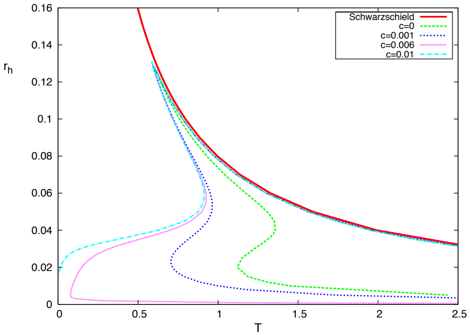
- Extremal limit, small c ($\Leftrightarrow$ GNS2): Regular solution
  - $a = b = m_\pi = 1$
  - Fixed α: Two solutions for sufficiently small c
  - $\Rightarrow$ unstable hairy BH might join unstable regular ($r_h = 0$) solution



- Extremal limit, small c: hairy BH solution
  - σ_h vs. r_h for $a = b = m_\pi = 1$, $\alpha = 0.05$
  - Upper branch solution exists for $r_h \to 0$ for small c, joins unstable regular solution $\Leftrightarrow$ GNS2
  - "phase transition" for some $c = c_{\rm cr}(\alpha)$
  - Here $c_{\rm cr}(0.05) \sim 0.006$



- Hairy Black Holes, Temperature
  - $a = b = m_\pi = 1$, $\alpha = 0.05$
  - Upper branch solution approaches $T = \infty$ again for $r_h \to 0$, for $c = 0$ and for $c < c_{\rm cr}$
  - approaches $T = 0$ at $r_h = r_h^{\rm ex}$ for $c > c_{\rm cr}$
  - Here $c_{\rm cr}(0.05) \sim 0.006$



- Qualitative understanding of r_h^{ex}
  - Expansion about r_h $\quad N \approx 0 + N_1(r - r_h) + \dots$

$$m \approx \frac{r_h}{2} + m_1(r - r_h) + \dots$$

$$\sigma \approx \sigma_h + \frac{\alpha^2 \sigma_h \sin^2 f_h J^2}{8 r_h H (r_h N_1)^2}(r - r_h) + \dots$$

$$f \approx f_h + \frac{r_h \sin f_h J}{2H(r_h N_1)}(r - r_h) + \dots$$

$$r_h N_1 = 1 - 2m_1 \equiv 1 - \frac{\alpha^2}{2}\left(m_\pi^2 r_h^2 (1 - \cos f_h) + 2a \sin^2 f_h + b\, r_h^{-2} \sin^4 f_h\right)$$

$$J \equiv m_\pi^2 r_h^4 + (b + 4a r_h^2) \cos f_h - b \cos(3 f_h)$$

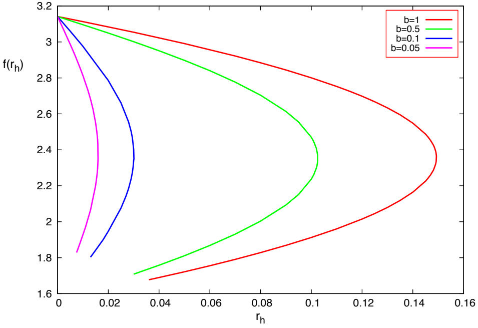
$$H \equiv a r_h^4 + 2b r_h^2 \sin^2 f_h + c \sin^4 f_h$$

- Possible sing. $r_h N_1 \to 0$ for small r_h
- For α too large: disaster
- For α small: may be avoided if $f_h \to \pi$ sufficiently fast for $r_h \to 0$: Happens for $c < c_{\rm cr}$
- $c > c_{\rm cr}$: $(r_h N_1)^{-1}$ is large close to sing.
- with $\sigma \approx \sigma_h + \sigma_1 (r - r_h) + \ldots$, where

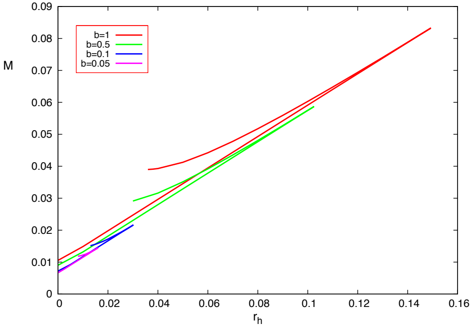
$$\sigma_1 = \frac{\alpha^2 \sigma_h \sin^2 f_h J^2}{8 r_h H (r_h N_1)^2}$$

  and $\sigma(r) < \sigma(\infty)$, large $r_h N_1$ compatible with b.c. $\sigma(\infty) = 1$ ONLY IF σ_h is small.
- $\Rightarrow$ b.c. $\sigma(\infty) = 1$ imposes small σ_h on numerical int., and $\sigma_h \to 0$ at $r_h = r_h^{\rm ex}$ before $r_h N_1 \to 0$

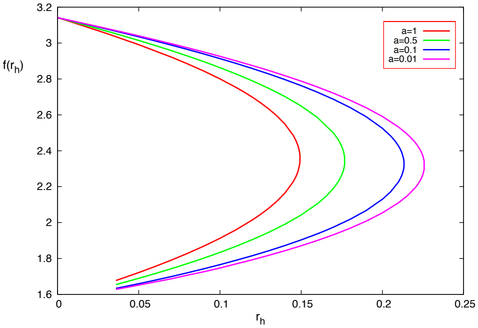
- Role of Skyrme term $\mathcal{L}_4$
  - f_h vs. r_h, for $a = c = m_\pi = 1$, $\alpha = 0.05$
  - $r_h^{\max}$ smaller for smaller b, $\lim_{b\to 0} r_h^{\max} = 0$
  - $\Rightarrow$ NO hairy black holes without the Skyrme term ($b = 0$)
  - although regular Skyrmion solutions exist



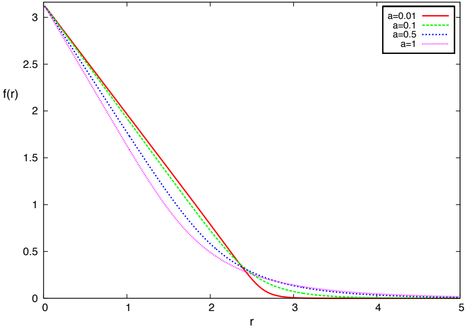
- Role of Skyrme term $\mathcal{L}_4$
  - Rescaled ADM mass $m(\infty)$ vs. r_h, for $a = c = m_\pi = 1$, $\alpha = 0.05$
  - Again, $\lim_{b\to 0} r_h^{\max} = 0$
  - As always, $M^{\mathrm{unst}}(r_h) > M^{\mathrm{st}}(r_h)$



- Role of n.l. Sigma-model term $\mathcal{L}_2$
  - f_h vs. r_h, for for $b = c = m_\pi = 1$, $\alpha = 0.05$
  - $r_h^{\max}(a)$ *increases* with decreasing a
  - Hairy BHs exist for $a = 0$



- Role of n.l. Sigma-model term $\mathcal{L}_2$
  - $f(r)$ vs. r, for for $b = c = m_\pi = 1$, $\alpha = 0.05$, and $r_h = 0.01$
  - For potential $\mathcal{U} = \mathcal{U}_\pi$, $a = 0$ solution is *compacton*
  - Compactons for less than quartic approach to vacuum



Summary of Results

- $\exists$ Skyrme-type models which
  - do possess flat space & self-gravitating top. solitons
  - but don't possess hairy Black Holes
  - Examples: BPS submodel (exact result)
    Model $\mathcal{L}_0 + \mathcal{L}_2 + \mathcal{L}_6$ (numerical)
- General condition for existence of hairy BHs
  - Presence of quartic (Skyrme) term $\mathcal{L}_4$
  - existence of top. soliton solutions (conjecture)
  - If applicable (for large B) to gravitating nuclear/hadronic matter, interesting repercussions: $\mathcal{L}_4$ might control formation of "hadron–black-hole" bound states ("neutron stars" with black hole cores)

- Influence of sextic term $\mathcal{L}_6$ on hairy BH solutions
  - For $c > c_{\text{cr}}(\alpha)$, unstable branch solutions $\nexists$ for $r_h \to 0$
  - Instead, at $r_h \searrow r_h^{\text{ex}}$, BH temperature $T \searrow 0$
  - Extremal ($T = 0$) hairy BH
  - At same $c_{\text{cr}}(\alpha)$, regular unstable sol. ceases to exist
- Future work:
  - Other potentials: results potential-independent? See e.g. GNS2
  - hairy BHs $\Leftrightarrow \mathcal{L}_4 \ldots$ deeper reason?
  - Higher-dim Skyrme models: which term required for hairy BHs?
  - Spinning stationary (Kerr-type) solutions: may induce hair even for $b = 0$?
  - Nonzero cosmological constant?
  - ...

Backup

Detailed calculation of curvature scalar:

$$R = -2\left(-2\frac{m'}{r^2} - \frac{m''}{r} + \frac{\sigma'}{\sigma}\left(-3\frac{m'}{r} - \frac{m}{r^2} + \frac{2}{r}\right) + \frac{\sigma''}{\sigma}\left(1 - 2\frac{m}{r}\right)\right)$$

$$\Rightarrow \quad -d^3x\sqrt{|g|}R = -d\Omega dr r^2 \sigma R = \dots$$

$$= 2d\Omega dr\left[-(rm\sigma)'' + (r\sigma')' - (rm\sigma')' + 2(m\sigma)' - 2m'\sigma\right].$$