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Topological Degree in Locating Homoclinic Structures for Discrete Dynamical Systems

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Abstract

A method of applying topological degree theory to analysis of chaotic behaviour of dynamical systems is described. The scheme combines one suggested by P. Zgliczyński with the method of topological shadowing. As an illustration a Henon mapping with a homoclinic tangency is considered.

Contents

1 Introduction

If $f : \mathbb{R}^d \mapsto \mathbb{R}^d$ is a continuous mapping, $U \subset \mathbb{R}^d$ is a bounded open set, $y \in \mathbb{R}^d$ does not belong to the image $f(\partial U)$ of the boundary ∂U of U, then the symbol $\deg(f, U, y)$ denotes the *topological degree* [3] of f at y with respect to U. If $0 \notin f(\partial U)$, then the number $\gamma(f, U) = \deg(f, U, 0)$ is well defined and it is called the *rotation of the vector field f at ∂U.* The properties of $\gamma(f, U)$ are described in detail in [11]. For an isolated root y of the equation $f(x) = 0$ the Kronecker index $\mathrm{ind}(a, f)$ is defined as the common value of the numbers $\gamma(f, B_a(\varepsilon))$. Here $\varepsilon > 0$ is sufficiently small, where $B_a(\varepsilon)$ denotes the open ball of radius ε, centred at a. The Kronecker index counts the generalized multiplicity of a root of the equation $f(x) = 0$; in this context, due to the Kronecker formula [11], $\gamma(f, U)$ can be interpreted as the algebraic number of roots of the equation $f(x) = 0$ located inside U.

The topological degree plays an important role in bifurcation analysis, see, e. g. [10] and the bibliography therein. Applications to analysis of complicated, chaotic-like behaviour are also well known, see references to Chapter 8 in [9], and especially the paper [19]. Recently an important new method for using the Kronecker index to locate topological Smale horseshoes in dynamical systems has been suggested by Zgliczyński [20] (earlier Conley Index theory was used in similar settings, see [2] and the bibliography therein). The Zgliczyński method was originally designed to prove that the shift on two elements is a factor of the dynamical system under consideration. It guarantees, when applicable, the existence of infinitely many periodic solutions, but does not, however, guarantee their instability.

In this paper we describe an alternative way of applying topological degree theory in analysis of systems with quasi-chaotic behaviour. This method is designed for locating homoclinic points and similar structures in multi-dimensional systems, and for analysis of higher dimensional perturbation of low dimensional systems. It combines the one suggested by P. Zgliczyński with the method of topological shadowing as described in [14]. Our method is similar to the Zgliczyński method in the two-dimensional case. Like the Zgliczyński method it is purely topological, and guarantees not only chaotic behaviour of the system itself, but also chaotic behaviour of all sufficiently small continuous perturbations. As a purely illustrative example, we discuss the robustness of quasi-chaotic and homoclinic structure in a Henon mapping with a homoclinic tangency.

2 Main result

2.1 Mappings compatible with topological Markov chains

For any positive integer m, let us denote by $\Omega(m)$ the totality of all bi–infinite sequences $\omega = \{\omega_i\}_{i=-\infty}^{\infty}$ with $\omega_i \in \{1, \dots, m\}$ for $i = 0$, ± 1, ± 2, …, and denote by $\sigma = \sigma_m$ the *(left) shift* on $\Omega(m)$ given by $\sigma_m(\omega) = \omega' = (\dots, \omega'_{-1}, \omega'_0, \omega'_1, \dots)$ where $\omega'_i = \omega_{i+1}$. Let $A = (a_{i,j})$, $i, j = 1, \dots, m$, be a square m-matrix whose entries are either zeros or ones, and denote by Ω_A the set $\{b \in \Omega(m) : a_{\omega_{i+1},\omega_i} = 1, i = 0, \pm 1,$ ± 2, …. The set Ω_A is shift invariant and the restriction σ_A of σ_m to Ω_A is the *topological Markov chain* with the matrix A.

Let f be a continuous mapping $\mathbb{R}^d \mapsto \mathbb{R}^d$. A *trajectory* of f (or, to be more precise, of the dynamical system generated by f) is a sequence $\mathbf{x} = \{x_i\}_{i=-i_-}^{\infty}$ satisfying $x_{i+1} = f(x_i)$, for $i = -i_-$, …, 0, 1, 2, …, where $0 \le i_- < \infty$ (note that $i_- = i_-(\mathbf{x})$ depends on a particular trajectory $\mathbf{x}$). σ_f is the left shift mapping naturally defined on the set $\mathrm{Tr}(f)$ of bi-infinite trajectories of f.

Let $\mathcal{X} = (X_1, \dots, X_m)$ be a finite family of compact connected subsets of $\mathbb{R}^d$. A continuous mapping f is $(\mathcal{X}, \sigma_A)$*–compatible* if there exists a mapping $\varphi : \Omega(A) \mapsto \mathrm{Tr}(f)$ which satisfies the following requirements:

(r1) the trajectory $x = \varphi(\omega)$ satisfies $x_i \in X_{\omega_i}$ for each $\omega \in \Omega_A$ and all integers i;

(r2) $\varphi\sigma_A = \sigma_f\varphi$: a shift of $\omega \in \Omega_A$ induces a shift of the trajectory $\varphi(\omega)$;

(r3) if $\omega \in \Omega_A$ is p-periodic, then the trajectory $\mathbf{x} = \varphi(\omega)$ is also p-periodic.

We assume neither the uniqueness of the mapping φ nor its continuity, so φ need not be a semi–conjugacy [9], p. 68. On the other hand, the subshift σ_A is a factor [9] of a restriction of the system f to some set $S \subset \bigcup X_i$, providing that f is $(\mathcal{X}, \sigma_A)$-compatible. The

$(\mathcal{X}, \sigma_A)$-compatible mappings have some features of chaotic behaviour if A has many ones, and if sufficiently many subfamilies of $\mathcal{X}$ have the empty intersections. For instance, from the definitions we have the following

Proposition 2.1. *(i) Let the sets X_i be disjoint and maximal eigenvalue λ of A be strictly greater than 1. Then the topological entropy $\mathcal{E}^{top}$ [9], p. 109, of f satisfies the inequality $\mathcal{E}^{top}(f) \geq \mathcal{E}^{top}(\sigma_A) =$* $\ln(\lambda)$. *(ii) Let $\bigcap_{i=1}^{m} X_i = \emptyset$, and the matrix A be transitive, in the sense that its power A^k has all positive entries. Then $\mathcal{E}^{top}(f)$ is positive.*

2.2 Principal theorem

Below we fix two positive integers d_u, d_s with $d_u + d_s = d$. Let V and W be bounded, open and convex product-sets

$$V = V^{(u)} \times V^{(s)} \subset \mathbb{R}^{d_u} \times \mathbb{R}^{d_s}, \quad W = W^{(u)} \times W^{(s)} \subset \mathbb{R}^{d_u} \times \mathbb{R}^{d_s},$$

satisfying the inclusions $0 \in V, W$ and let $g : \overline{V} \mapsto \mathbb{R}^{d_u} \times \mathbb{R}^{d_s}$ be a continuous mapping. It is convenient to treat g as the pair $(g^{(u)}, g^{(s)})$ where $g^{(u)} : V \mapsto \mathbb{R}^{d_u}$ and $g^{(s)} : V \mapsto \mathbb{R}^{d_s}$. The mapping g is (V, W)-*hyperbolic*, if the equations

$$\begin{gathered} g^{(u)}\left(\partial V^{(u)} \times \overline{V}^{(s)}\right) \bigcap \overline{W}^{(u)} = \emptyset, \\ g(\overline{V}) \bigcap \left(\overline{W}^{(u)} \times (\mathbb{R}^{d_s} \setminus W^{(s)})\right) = \emptyset, \end{gathered} \tag{2.1}$$

hold, and

$$\deg(g^{(u)}(\cdot, 0), V^{(u)}, 0) \neq 0. \tag{2.2}$$

Here $\overline{S}$ denotes the closure of a set S. The first relationship (2.1) means geometrically that the image of the 'u-boundary' $\partial V^{(u)} \times \overline{V}^{(s)}$ of V does not intersect the infinite cylinder $C = \overline{W}^{(u)} \times \mathbb{R}^{d_s}$; analogously, the second part of (2.1) means that the image of the whole set $g(V)$ can intersect the cylinder C only by its central fragment $\overline{W}^{(u)} \times W^{(s)}$. Thus the first equation (2.1) means that the mapping expands in a rather weak sense along the first coordinate in the Cartesian product $\mathbb{R}^{d_u} \times \mathbb{R}^{d_s}$, whereas the second one confers a type of contraction along the second coordinate (the indices '(u)' and '(s)' refer to the adjectives 'stable' and 'unstable'). Figure 1 at the end of this section illustrates the geometrical meaning of relationships (2.1) in the two-dimensional case.

Theorem 1. *Let A be a square $m-$matrix whose entries are either zeros or ones, $h_i : \mathbb{R}^{d_u} \times \mathbb{R}^{d_s} \mapsto \mathbb{R}^d$ be homeomorphisms, and V_i be bounded, open and convex product sets. Suppose that $g_{i,j} = h_j^{-1} f h_i$ is $(V_i, V_j)-$hyperbolic whenever $a_{i,j} = 1$. Then there exist compact sets $X_i \subset h_i(V_i)$ such that f is $(\mathcal{X}, \sigma_A)$-compatible.*

Proof is relegated to Section 4.

Recall, that a trajectory $\mathbf{x} = \{x_i\}_{i=-\infty}^{\infty}$ of a continuous bounded mapping $f : \mathbb{R}^d \mapsto \mathbb{R}^d$ is called *homoclinic* if its elements are not all identical and there exists a point $x_* \in \mathcal{X}$ such that $\lim_{i\to-\infty} x_i = \lim_{i\to\infty} x_i = x_*$. The point x_* is a *homoclinic fixed point.*

Corollary 2.1. *Let h_i, $i = 1, \dots, m$, be homeomorphisms, V_i be bounded open product sets and*

$$\bigcap_{i=1}^{m} h_i(V_i) = \emptyset. \tag{2.3}$$

Let the mappings $g_{i,j} = h_j^{-1} f h_i$ be $(V_i, V_j)-$hyperbolic for the pairs (1,1), (m,1) and $(i, i+1), 1 \le i < m\}$. Suppose, finally, that there exists at most one bi-infinite trajectory in V_1. Then there exists a unique homoclinic point $x_* \in h_1(V_1)$.

P r o o f. Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}. \tag{2.4}$$

The conditions of the theorem are met with respect to this matrix, and thus the mapping f is $(\mathcal{X}, \sigma_A)-$compatible for some compact sets $X_i \subset h_i(V_i)$.

Let us consider the symbolic sequence ω^* defined by $\omega_i^* \equiv 1$. The theorem and the requirement (r2) guarantee that there exists a fixed point x_* which belongs to $h_1(V_1)$.

Now, consider the symbolic sequence ω defined by $\omega_i = i$, for $i = 1, \dots, m$, and $\omega_i = 1$ otherwise. By the theorem there exists a trajectory $\mathbf{x}$ satisfying $x_i \in X_{\omega_i}$ for all i. This trajectory is not equal to x_* for all i, because of (2.3). It remains to establish the equalities $\lim_{i\to-\infty} x_i = \lim_{i\to\infty} x_i = x_*$. Suppose that they are wrong: let, for instance, $\lim_{i\to\infty} x_i \neq x_*$. Then there exists a sequence $j(i) \to \infty$ satisfying $|x_{j(i)} - x_*| \geq \varepsilon$ for some strictly positive ε. Taking a coordinate-wise limit of the sequence of trajectories $\mathbf{x}^{(k)} = (\ldots, x_{-1}^{(k)}, x_0^{(k)}, x_1^{(k)}, \ldots)$, $\quad k = 1, 2, \ldots$, where $x_i^{(k)} = x_{i-j(k)}$, we obtain a trajectory $\mathbf{y}$ which satisfies $|y_0 - x_*| \geq \varepsilon > 0$, is contained in $h_1(V_1)$, and is not identically equal to x_*. It contradicts the last assumption of the corollary. ∎

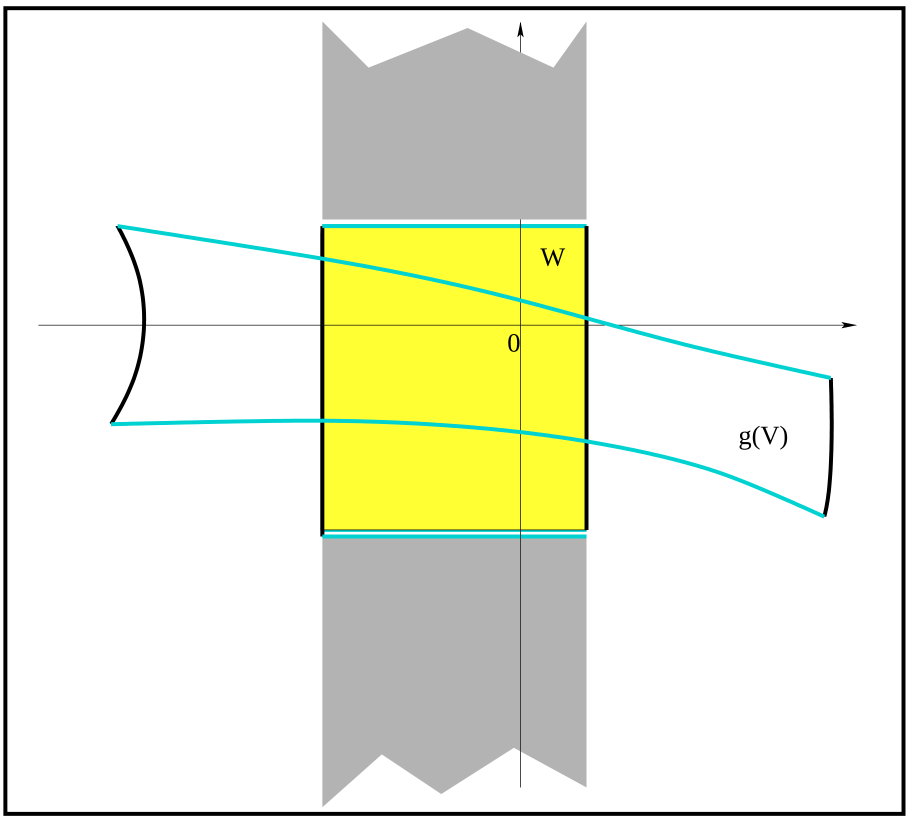

Figure 1: Horizontal axes represents $\mathbb{R}^{d_u}$ and the vertical one represents $\mathbb{R}^{d_s}$ for $n_u = n_s = 1$. The lightly shaded rectangle represents $W = W^{(u)} \times W^{(s)}$. The darker shadowed is a part of the infinite cyllinder $C = \overline{V} \times \mathbb{R}^{(d_s}$. This cyllinder can not be intersected by the image $g(V)$; the images of $\partial V^{(u)} \times \overline{V}^{(s)}$ also can not intersect W. The deformed quadri-lateral represents an admissible location of $g(V)$ (the black part of its border are images of the sets $\partial V^{(u)} \times V^{(s)}$, whereas images of $V^{(u)} \times \partial V^{(s)}$ are gray.)

To conclude this section we note that applying the theorem and the corollary are simplified significantly if $d_u = 1$. In this case the mapping $g_{i,j}^{(u)}(0, x^{(u)})$ is one dimensional, $V_i^{(u)}$ is an interval (α_i, β_i) with $\alpha_i\beta_i < 0$, and verifying the inequality (2.2) is trivial: *The inequality (2.2) holds if and only if* $g_i^{(u)}(0, \alpha_i)g_i^{(u)}(0, \beta_i) < 0$; *moreover, in this case*

$$\deg(g_i^{(u)}(\cdot, 0), V_i^{(u)}, 0) = \text{sign}\,(g_{i,j}^{(u)}(0, \beta_i)). \tag{2.5}$$

3 Discussion

3.1 Chaotic behaviour

Important attributes of chaotic behaviour include sensitive dependence on initial conditions, an abundance of periodic trajectories and an irregular mixing effect describable informally by the existence of a finite number of separated subsets $U_1, \ldots, U_m$ of $\mathbb{R}^d$ which can be visited by trajectories of some fixed iterate f^k of f in any prescribed order. Let $\mathcal{U} = \{U_1, \ldots, U_m\}$, $m > 1$, be a family of disjoint subsets of $\mathbb{R}^d$ and let us denote the set of one-sided sequences $\omega = \omega_0, \omega_1, \ldots,$ by Ω_m^R. Sequences in Ω_m^R will be used to prescribe the order in which sets U_i are to be visited. For $x \in \bigcup_{i=1}^m U_i$ we denote by $I(x)$ the number i satisfying $x \in U_i$.

A mapping f is called $(\mathcal{U}, k)$-*chaotic* (k is a positive integer) if there exists a compact f-invariant set $S \subset \bigcup_i U_i$ with the following properties:

(p1) for any $\omega \in \Omega_m^R$ there exists $x \in S$ with $f^{ik}(x) = (f^p)^i \in U_{\omega_i}$ for $i \geq 1$;

(p2) for any p-periodic sequence $\omega \in \Omega_m^R$, there exists a pk-periodic point $x \in S$ with $f^{ik}(x) \in U_{\omega_i}$;

(p3) for each $\eta > 0$ there exists an uncountable subset $S(\eta)$ of S, such that the simultaneous relationships

$$\limsup_{i\to\infty} \left| I(f^{ik}(x)) - I(f^{ik}(y)) \right| \geq 1,$$

$$\liminf_{i\to\infty} \left| f^{ik}(x) - f^{ik}(y) \right| < \eta$$

hold for all $x, y \in S(\eta), x \neq y$.

The above defining properties of chaotic behaviour are similar to those in the Smale transverse homoclinic trajectory theorem with an important difference being that we do not require the existence of an invariant Cantor set. Instead, the definition includes property (p2), which is usually a corollary of uniqueness, and (p3) which is a form of sensitivity and irregular mixing as in the Li–Yorke definition of chaos, with the subset $S(\eta)$ corresponding to the Li–Yorke scrambled subset S_0. A similar definition was used previously in [4]. Note also that the one-sided left shift σ_m^+ is a factor of the restriction $f^k|_S$ ([9], p. 68). Theorem 1 implies

Proposition 3.1. *Let $\mathcal{X} = (X_1, \ldots, X_m)$ be a family of compact sets and let the matrix A be k-transitive (that is A^k have strictly positive entries). Suppose that the mapping f is $(\mathcal{X}, \sigma_A)$ compatible and suppose that the family $\mathcal{U}$ of connected components of the union set $\mathbf{U} = \bigcup_{i=1}^m X_i$ has more than one element. Then the mapping f is $(\mathcal{U}, k)$-chaotic.*

P r o o f. Let the mapping $\varphi : \Omega_A \mapsto \mathrm{Tr}(f)$ satisfy the requirements (r1)–(r3) from the definition of $(\mathcal{X}, \sigma_A)$ compatibility. Let us define S as closure of the set $S_0 = \{(\varphi(\omega))_i : \omega \in \Omega_A, i = 0, \pm1, \pm2, \ldots\}$. The properties (p1) and (p2) hold by construction, and we should only take care of the property (p3).

Let Ω_+ be the set of restrictions of sequences from Ω_A to non-negative indices. Consider the equivalence relation E on Ω_+ defined by: $E(\omega, \omega')$ is true if and only if $X_{\omega_{ik}}$ and $X_{\omega'_{ik}}$ belong to the same connected component of $\mathbf{U}$ for all sufficiently large i. Denote the set of equivalence classes by $\mathcal{T}$ and note that the set $\mathcal{T}$ has the power of continuum (because A is transitive). Choose a single element $\omega_+(T)$ from each equivalence class in $\mathcal{T}$, and denote by $\omega(T)$ a sequence from Ω_A which coincides with $\omega_+(T)$ for non-negative indices. The set $S_* = \{\varphi(\omega) : \omega(T), T \in \mathcal{T}\}$ is a subset of S and the first inequality in (p3) holds for any two different elements of S_*. It remains to use the following general statement.

Lemma 3.1. *Let (M, ρ) be a compact metric space, g be a mapping $M \mapsto M$, and let S be a subset of M having the power of continuum. Then for each $\eta > 0$ there exists a subset $S(\eta)$ of S with the power of the continuum such that* $\liminf_{i\to\infty} \rho\left(g^i(x), g^i(y)\right) < \eta$ *for any x, $y \in S(\eta)$.*

P r o o f. By compactness of (M, ρ) there exists a finite partition $\mathcal{P}$ of M such that $\mathrm{diam}(P) < \eta$ for each $P \in \mathcal{P}$. Two elements x, $y \in$

S are said to be *connected*, if there exist arbitrarily large j for which $g^j(x)$ and $g^j(y)$ belong to the same subset from the partition. Since there are connected elements in every set which contains more than $\#(\mathcal{P})$ elements, there are connected elements in every denumerable set. The assertion of the Lemma will hold if a subset $S(\eta) \subseteq S$ of pairwise connected elements which has the power of the continuum can be constructed. That this can be done follows by an application of a transfinite analogue of the Ramsey Complete Graph Theorem (cf. [5], page 608, Theorem 5.23: *If Γ is a graph of power m, where m is a transfinite cardinal, and if every denumerable subset of G two connected elements, then Γ contains a complete graph of power m*). This completes the proof of the lemma, and, hence, the proof of the proposition as well. □

Theorem 1 and the proposition above imply

Corollary 3.1. *Let a matrix A be k-transitive and let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a continuous mapping. Let there exist homeomorphisms h_i and product sets V_i such that $h_j^{-1} f h_i$ is (V_i, V_j)-hyperbolic if $a(i,j) = 1$, and let the family $\mathcal{U}$ of connected components of the union set $\mathbf{U} = \bigcup h_i(V_i)$ have more than one element. Then any mapping $\tilde{f}$, which is sufficiently close to f in the uniform metric, is $(\mathcal{U}, k)$-chaotic.*

Let us formulate another corollary of Proposition 3.1 and Theorem 1. A continuous mapping f is (ε, k)–chaotic in a neighborhood of $\mathcal{X} \subset \mathbb{R}^d$ if for each finite subset $\mathcal{X}_* = \{x_1, \ldots, x_m\}$ of $\mathcal{X}$ with $\min_{i \neq j} |x_i - x_j| \geq 2\varepsilon$ the mapping f is (B)-chaotic where $\mathcal{B}$ is the family of (disjoint) balls $B_{x_i}(\varepsilon)$ The minimal $\varepsilon_0 \geq 0$ with the property that for each $\varepsilon > \varepsilon_0$ the system f is $(\varepsilon, k)-$chaotic for an appropriate k is called the *chaos threshold* of the system f with respect to $\mathcal{X}$. The chaos threshold characterizes accuracy of measurements for which the behaviour of the system in the vicinity of the subset $\mathcal{X}$ appears chaotic if the time lapse between subsequent measurements is sufficiently large.

Corollary 3.2 *Suppose that there exists a transitive matrix A, homeomorphisms h_i and product sets V_i such that $h_j^{-1} f h_i$ is $(V_i, V_j)-$hyperbolic if $a(i,j) = 1$. Suppose that for some positive Δ and r the inclusions $\mathcal{X} \subset \bigcup B_{h_i(0)}(\Delta)$ and $h(V_i) \subset B_{h_i(0)}(r)$ are valid. Then the chaos threshold of f with respect to $\mathcal{X}$ is not greater than $\Delta + r$.*

3.2 Smooth systems

One more important attribute of chaotic behaviour is the abundance of *unstable* periodic trajectories. The method introduced above could also be useful in the investigation of this property, especially for dynamical systems generated by a smooth function f.

Suppose that the condition of Theorem 1 are met. Then to each periodic symbolic sequence $\omega \in \Omega_A$, we can associate the number

$$\gamma(\omega) = (-1)^{d_u} \prod_{i=1}^{p} \deg(g^{(u)}_{\omega_i,\omega_{i-1}}(\cdot, 0), W^{(u)}_{\omega_{i-1}}, 0).$$

Denote by P_- the set of all periodic sequences which satisfy $\gamma(\omega) < 0$.

Lemma 3.2 *The mapping φ can be chosen such that $\varphi(\omega)$ is not asymptotically stable whenever $\gamma(\omega) < 0$.*

The proof will be given simultaneously with that of the theorem.

Proposition 3.2 *Let f be smooth. Then for a small generic perturbation $\tilde{f}$ the mapping $\tilde{\varphi}$ could be chosen such that $\varphi(\omega)$ is exponentially unstable if $\gamma(\omega) < 0$.*

P r o o f. It is sufficient to combine the previous proposition with Theorem β, [18], p. 177. This theorem guarantees that hyperbolic endomorphisms are generic; in particular, they are generic endomorphisms which have only finite number of periodic orbits in any bounded subset, and all these orbits are exponentially stable or exponentially unstable. □

To conclude this section, we mention how our methods could be used in line with the classical Sullivan-Shub result [19].

Proposition 3.3. *Let $V = V^{(u)} \times V^{(s)}$ be a product set. Let a smooth mapping f be (V, V)-hyperbolic, and suppose that the inequality*

$$|\deg(f^{(u)}(\cdot, 0), V^{(u)}, 0)| > 1 \tag{3.1}$$

holds. Then f has infinitely many periodic orbits in V.

P r o o f. Let us consider the set $S = S(V, V)$. By Lemma 4.1 (which will be proved in the next section) there exists a mapping g which coincides with f on S, has no periodic points outside of S, and satisfies

$$\deg\left(id - g^n, V, 0\right) = (-1)^{d_u} \left(\deg(f^{(u)}(\cdot, 0), V^{(u)}, 0)\right)^n \tag{3.2}$$

for all positive integer n. It remains to refer to [19] (see also [9], p. 323). □

3.3 Small perturbations

It is important to note that the set of functions f such that the conditions of the above theorem hold (for given families of homeomorphisms h_i and sets V_i and a given matrix A) is open with respect to the uniform metric. Thus if the theorem is applicable to some function f, then it is also applicable to any sufficiently small uniform perturbation $\tilde{f}$.

In this subsection we suppose that f is a continuous function; A is a square m-matrix whose entries are either zeros or ones; $h_i : \mathbb{R}^{d_u} \times \mathbb{R}^{d_s} \mapsto \mathbb{R}^d$ are homeomorphisms, and V_i are bounded, open and convex product sets. We suppose that $g_{i,j} = h_j^{-1} f h_i$ is (V_i, V_j)-hyperbolic whenever $a_{i,j} = 1$. We define the numbers

$$\begin{aligned} \chi_{i,j}^{(u)} &= \inf\left\{ |fh_i(x) - h_j(y)| : x \in \partial V_i^{(u)} \times \overline{V_i}^{(s)},\ y \in \overline{V_j^{(u)}} \times \mathbb{R}^{d_s} \right\}, \\ \chi_{i,j}^{s} &= \inf\left\{ |fh_i(x) - h_j(y)| : x \in \overline{V_i},\ y \in \overline{V_j}^{(u)} \times (\mathbb{R}^{d_s} \setminus V_j^{(s)}) \right\}. \end{aligned}$$

whenever $a_{i,j} = 1$. Define also

$$\chi = \max_{a_{i,j}=1} \{\chi_{i,j}^u, \chi_{i,j}^u\}. \tag{3.3}$$

Let now $\widehat{d}$ be a positive integer and $\widehat{d}_s = d_s + \widehat{d}$. Consider a continuous function

$$F : \mathbb{R}^d \times \mathbb{R}^{\widehat{d}} \mapsto \mathbb{R}^d \times \mathbb{R}^{\widehat{d}}.$$

We will treat it as the pair (F_1, F_2) where $F_1 : \mathbb{R}^d \times \mathbb{R}^{\widehat{d}} \mapsto \mathbb{R}^d$ and $F_2 : \mathbb{R}^d \times \mathbb{R}^{\widehat{d}} \mapsto \mathbb{R}^{\widehat{d}}$. Suppose that the following estimates are valid:

$$|F_1(x, y) - f(x)| \le \varepsilon_1 + c_1|y|, \quad |F_2(x, y)| \le \varepsilon_2 + c_2|y|.$$

Here $\varepsilon_1, \varepsilon_1, c_1, c_2 > 0$, $c_2 < 1$ and $\varepsilon_1 + c_2\varepsilon_2/(1 - c_2) < \chi$. Introduce the homeomorphism $\widehat{h}_i = (h_i, id)$, which will be treated below as mappings from $\mathbb{R}^{(d_u)} \times \mathbb{R}^{(\widehat{d}_s)} \mapsto \mathbb{R}^{\widehat{d}}$, and the product sets $\widehat{V}_i = V_i^{(u)} \times \widehat{V}_i^{(s)}$ with $\widehat{V}_i^{(s)} = V_i^{(s)} \times \mathcal{B}_\delta^{(\widehat{d})}$. Here $\mathcal{B}^{(\widehat{d})}$ is the open $\varepsilon_2/(1 - c_2)$-ball in $\mathbb{R}^{\widehat{d}}$ centred at zero and $\delta = \varepsilon_2/(1 - c_2)$.

Proposition 3.4 *The mapping* $G_{i,j} = \widehat{h}_j^{-1} F \widehat{h}_i$ *is* $(\widehat{V}_i, \widehat{V}_j)$*-hyperbolic if* $a_{i,j} = 1$.

P r o o f. Straightforward calculation. □

4 Proof of Theorem 1 and Lemma 3.2

4.1 Auxiliary lemma

Let

$$W_i = W_i^{(u)} \times W_i^{(s)} \subset \mathbb{R}^{d_u} \times \mathbb{R}^{d_s}, \quad i = 0, \ldots, n,$$

be convex open and bounded product sets, and suppose that the mappings

$$g_i : \overline{W}_{i-1}^{(u)} \times \overline{W}_{i-1}^{(s)} \mapsto \mathbb{R}^{d_u} \times \mathbb{R}^{d_s}, \qquad i = 1, \ldots, n,$$

are $(W_{i-1}, W_i)-$hyperbolic. In particular,

$$g_i^{(u)}(\partial W_{i-1}^{(u)} \times \overline{W}_{i-1}^{(s)}) \bigcap \overline{W}_i^{(u)} = \emptyset$$

$$\tag{4.1}$$

$$g_i(\overline{W}_{i-1}) \bigcap \left(\overline{W}_i^{(u)} \times (\mathbb{R}^{d_s} \setminus W_i^{(s)}) \right) = \emptyset,$$

and

$$\deg(g_i^{(u)}(\cdot, 0), V_{i-1}^{(u)}, 0) \neq 0 \tag{4.2}$$

for $i = 1, \ldots, n$.

Let us now define the auxiliary mappings q_i, $i = 1, \ldots, n$, by

$$q_i : (\mathbb{R}^{d_u} \times \mathbb{R}^{d_s}) \mapsto (\mathbb{R}^{d_u} \times \mathbb{R}^{d_s}), \quad i = 1, \ldots, n.$$

By the second equation (4.1) we can choose compact convex sets $T_i^{(s)} \subset \mathbb{R}^{d_s}$, $i = 1, \ldots, n$, satisfying

$$g_i(W_{i-1}) \bigcap \left(W_i^{(u)} \times \mathbb{R}^{d_s} \right) \subset W_i^{(u)} \times T_i^{(s)}.$$

Let $\mathbb{R}^{d_u} \times \mathbb{R}^{d_s} = \mathbb{R}^d$ be endowed with the standard Euclidean metric. We define the stable component $q_i^{(s)}(y)$ as the projection (the nearest point) of $g_i^{(s)}(y)$ on $T_i^{(s)}$, and define the unstable component $q_i^{(u)}(y)$ by $q_i^{(u)}(y) = g_i^{(u)}(y)$ where y is the projection of y on $\overline{W}_{i-1}$. Let us define the iterated mappings Q_i by $Q_0 = id$ and $Q_i = q_i(Q_{i-1})$ for $i = 1, \ldots, n$ (by 'id' we denote the identity mapping). Let us introduce the sets

$$S_i = \left\{ y \in W_i : g_{i+1}^{(u)}(y) \in W_{i+1}^{(u)} \right\}. \tag{4.3}$$

These sets are nonempty open sets by (4.1) and (4.2) . It is important that q_i coincides with g_i on S_{i-1}:

$$q_i(y) = g_i(y), \quad y \in S_{i-1}, \ i = 1, \ldots, n, \tag{4.4}$$

and we can rewrite (4.3) as

$$S_i = \left\{ y \in W_i : q^{(u)}_{i+1}(y) \in W^{(u)}_{i+1} \right\}.$$

Lemma 4.1.

(a) The simultaneous inclusions $Q^{(u)}_n(y) \in W^{(u)}_n$, $y^{(s)}_0 \in W^{(s)}_0$ *imply* $Q_i(y) \in S_i$ *for* $i = 0, 1, \dots, n-1$.

(b) $\deg \left(Q^{(u)}_n(\cdot, 0), W^{(u)}_0, 0 \right) = (-1)^{d_s} \prod_{i=1}^{n} \deg(q^{(u)}_i(\cdot, 0), W^{(u)}_{i-1}, 0)$.

(c) If $W_n = W_0$, *then* $\deg(id - Q_n, W_0, 0) = (-1)^{d_u} \prod_{i=1}^{n} \deg(q^{(u)}_i(\cdot, 0), W^{(u)}_{i-1}, 0)$.

P r o o f. (a) By construction the functions q_i satisfy the relationships:

$$q^{(u)}_i \left(\left(\mathbb{R}^{d_u} \setminus W^{(u)}_{i-1} \right) \times \mathbb{R}^{d_s} \right) \subset \mathbb{R}^{d_u} \setminus \overline{W}^{(u)}_i, \quad q^{(s)}_i(\mathbb{R}^d) \subset T^{(s)}_i \tag{4.5}$$

for $i = 1, \dots, n$. The first inclusion in (a) and the first inclusion (4.5) imply

$$Q^{(u)}_i(y) \in W^{(u)}_i, \quad i = 0, \dots, n, \tag{4.6}$$

by induction. On the other hand, the second inclusion (a) and the second inclusion (4.5) imply $Q^{(s)}_i(y) \in W^{(s)}_i$, $i = 0, \dots, n$. Thus, taken into account (4.6),

$$Q_i(y) \in W_i, \quad i = 0, \dots, n. \tag{4.7}$$

Assertion (a) follows from (4.6), (4.7) and (4.3).

It remains to establish (b) and (c). The proofs of these two assertions are similar, with the later being a bit more interesting. So we will prove in detail only (c).

Let us define the mappings $q_{i,\vartheta} : \mathbb{R}^d \times \mathbb{R}^1 \mapsto \mathbb{R}^d$ for $0 \le \vartheta \le 1$, $i = 1, \dots, n$, by

$$q_{i,\vartheta}(y) = \left(q^{(u)}_i(y^{(u)}, (1-\vartheta)y^{(s)}), q^{(s)}_i((1-\vartheta)y^{(u)}, y^{(s)}) \right).$$

The mappings $q_{i,\vartheta}$, $1 \le \vartheta \le 1$ satisfy the inclusions

$$q^{(u)}_{i,\vartheta} \left(\left(\mathbb{R}^{d_u} \setminus W^{(u)}_{i-1} \right) \times \mathbb{R}^{d_s} \right) \subset \mathbb{R}^{d_u} \setminus \overline{W}^{(u)}_i, \quad q^{(s)}_{i\vartheta}(\mathbb{R}^d) \subset T^{(s)}_i \tag{4.8}$$

together with q_i, see (4.5). Therefore, any fixed point $y \in \overline{W}_0$ of $Q_{n,\vartheta}$ belongs to W_0. Thus, the deformation $Q_{n,\vartheta} - id$ is nonsingular on ∂W_0, that is

$$Q_{n,\vartheta}(y) \neq y \quad \text{for} \quad y \in \partial W_0.$$

This deformation is also continuous and compact. Therefore, $Q_{n,0} - id$ and $Q_{n,1} - id$ are *homotopic* ([11], p. 93) on ∂D_0 and

$$\deg(Q_{n,0} - id, W_0, 0) = \deg(Q_{n,1} - id, W_0, 0) \tag{4.9}$$

by Theorem 20.1 [11], p. 100. On the other hand, the mapping $Q_{n,1} - id$ is the *direct sum*, [11], p.117 of the mapping $Q^{(u)}_{n,1}(0, y^{(u)}) - id$ on $W^{(u)}_0 \subset \mathbb{R}^{d_u}$ and the mapping $Q^{(s)}_{n,1}(y^{(s)}, 0) - id$ on $W^{(s)}_0 \subset \mathbb{R}^{d_s}$ in $\mathbb{R}^{d_s}$:

$$Q_{n,1}(\cdot) - id = (Q^{(u)}_{n,1}(\cdot, 0) - id) \oplus (Q^{(s)}_{n,1}(0, \cdot) - id).$$

Thus, Theorem 22.4 [11], p. 117 implies that

$$\deg\left(Q_{n,1} - id, W_0, 0\right) =$$

$$= \deg\left(Q^{(u)}_{n,1}(\cdot, 0) - id, W^{(u)}_0, 0\right) \deg\left(Q^{(s)}_{n,1}(0, \cdot) - id, W^{(s)}_0, 0\right).$$

The last equality can be rewritten as

$$\deg\left(Q_{n,0} - id, W_0, 0\right) =$$

$$= \deg\left(Q^{(u)}_{n,1}(\cdot, 0) - id, W^{(u)}_0, 0\right) \deg\left(Q^{(s)}_{n,1}(0, \cdot) - id, W^{(s)}_0, 0\right)$$

by (4.9) . Since

$$\deg\left(Q^{(s)}_{n,1}(0, \cdot) - id, W^{(s)}_0, 0\right) =$$

$$= (-1)^{d_s} \deg\left(id - Q^{(s)}_{n,1}(0, \cdot), W^{(s)}_0, 0\right) = (-1)^{d_s}$$

by the first inclusion (4.1) and Theorem 21.5, [11], p.108, it can be rewritten further as

$$\deg\left(Q_{n,0} - id, W_0, 0\right) = (-1)^{d_s} \deg\left(Q^{(u)}_{n,1}(\cdot, 0) - id, W^{(u)}_0, 0\right). \tag{4.10}$$

On the other hand, $Q^{(u)}_{n,1}(w) \notin W^{(u)}_n = W^{(u)}_0$ for $w \in \partial W^{(u)}_0$, and so the vectors $Q^{(u)}_{n,1}(w, 0) - w$ and $Q^{(u)}_{n,1}(w, 0)$ do not point in opposite directions for $w \in \partial W^{(u)}_0$, that is $Q^{(u)}_{n,1}(w, 0) - id = \mu Q^{(u)}_{n,1}(w, 0)$ does not hold for any $\mu \geq 0$. (Indeed, otherwise $Q^{(u)}_{n,1}(w, 0) = w/(1 + \mu)$;

that contradicts the first inclusion (4.8) because $0 \in W_0^{(u)}$). Therefore $Q_{n,1}^{(u)}(w,0) - w$ and $Q_{n,1}^{(u)}(w,0)$ are homotopic on $\partial D_0^{(u)}$ by Theorem 2.1 [11], p. 4, and, further, the equation

$$\deg\left(Q_{n,1}^{(u)}(\cdot,0) - id, W_0^{(u)}, 0\right) = \deg\left(Q_{n,1}^{(u)}(\cdot,0), W_0^{(u)}, 0\right).$$

holds by Property 1 [11], p.5. Now (4.10) implies

$$\deg\left(Q_{n,0} - id, W_0, 0\right) = (-1)^{d_s}\deg\left(Q_{n,1}^{(u)}(\cdot,0), W_0^{(u)}, 0\right).$$

On the other hand,

$$\deg\left(Q_{n,0} - id, W_0, 0\right) = (-1)^{d}\deg\left(id - Q_{n,0}, W_0, 0\right)$$

(for instance, again by the product formula (7.6), [11]). The last two equations imply

$$\deg\left(id - Q_{n,0}, W_0, 0\right) = (-1)^{d_u}\deg\left(Q_{n,1}^{(u)}(\cdot,0), W_0^{(u)}, 0\right).$$

It remains to establish that

$$\deg\left(Q_{n,1}^{(u)}(\cdot,0), W_0^{(u)}, 0\right) = \prod_{i=1}^{n}\deg\left(q_i^{(u)}(\cdot,0), W_{i-1}^{(u)}, 0\right).$$

Since $q_i^{(u)}(0, y^{(u)}) = q_{i,1}^{(u)}(0, y^{(u)})$ for $y^{(u)} \in \partial W_i^{(u)}$, we have to establish the equality

$$\deg\left(Q_{n,1}^{(u)}, W_0^{(u)}, 0\right) = \prod_{i=0}^{n}\deg\left(q_{i,1}^{(u)}(\cdot,0), W_{i-1}^{(u)}, 0\right). \tag{4.11}$$

Let $\overline{W}_i^{(u)}$ be contained in an open ball B_i, $i = 1, \ldots, n$. The values of the mappings $q_{i,1}^{(u)}(w)$ belong to the closed and bounded set $Z_i = \overline{B_i} \setminus W_i^{(u)}$ for $w \in \partial W_{i-1}^{(u)}$. Indeed, this assertion is true for $i = 1$ by the second equality (4.1), and can be proved by induction for other i. The set $\mathbb{R}^{d_u} \setminus Z_i$ consists only of two connected components with 0 being contained in the bounded component. The mapping $q_{i,1}^{(u)}$ is nonsingular and continuous on Z_i. Thus, the product formula for rotations (see Theorem 7.2 and the formula (7.6), [11], p. 18) implies

$$\deg(Q_{i+1,1}^{(u)}, W_0^{(u)}, 0) = \deg(Q_{i,1}^{(u)}, W_0^{(u)}, 0) \cdot \deg(q_{i+1,1}^{(u)}, W_i^{(u)}, 0),$$

$$i = 1, \ldots, n-1.$$

Since $Q_{1,1}^{(u)} = q_{1,1}^{(u)}$, the 'base equation' $\deg(Q_{1,1}^{(u)}, W_0^{(u)}, 0) = \deg(q_{1,1}^{(u)}, W_0^{(u)}, 0)$ also holds, and (4.11) follows inductively. The lemma is proved. □

Denote $Y_i = \overline{S}_i \bigcap g_i(\overline{S}_{i-1})$, $i = 1, \ldots, n-1$. The sets Y_i are compact subsets of W_i by (4.1). In the case $W_0 = W_n$ it is convenient to define also $Y_0 = \overline{S}_0 \bigcap g_i(\overline{S}_{n-1})$.

Corollary 4.1. *There exists a sequence satisfying* $y_i = g_i(y_{i-1})$, $i = 1, \ldots, n$, *and* $y_i \in Y_i$, $i = 1, \ldots, n-1$. *If additionally* $W_n = W_0$, *then there exists a sequence* $y_i \in W_i$ *satisfying additionally the equality* $y(n) = y_0$ *and the inclusion* $y_0 \in Y_0$.

P r o o f. Firstly we prove the equalities

$$\deg(g_i^{(u)}(\cdot, 0), W_{i-1}^{(u)}, 0) = \deg(q_i^{(u)}(\cdot, 0), W_{i-1}^{(u)}, 0). \tag{4.12}$$

Denote

$$S_{i,0} = \left\{ w \in W^{(u)} : g_i(0, w) \in W_i^{(u)} \right\} = S_i \bigcap (0 \times \mathbb{R}^{d_u}).$$

By the definition, neither $g_i^{(u)}(0, w)$, nor $q_i^{(u)}(0, w)$ have zeros outside $S_{i,0}$. Thus

$$\deg(g_i^{(u)}(\cdot, 0), W_{i-1}^{(u)}, 0) = \deg(g_i^{(u)}(\cdot, 0), S_{i-1,0}, 0)$$

and

$$\deg(q_i^{(u)}(\cdot, 0), W_{i-1}^{(u)}, 0) = \deg(q_i^{(u)}(\cdot, 0), S_{i-1,0}, 0).$$

On the other hand, $g_i^{(u)}(\cdot, 0)$ coincides with $q_i^{(u)}(\cdot, 0)$ in $S_{i,0}$ by (4.4), and (4.12) is proved.

By the assertions (b) and (c) of the lemma, the simultaneous relationships (4.2), (4.12) imply $\deg\left(Q_n(\cdot, 0), W_0^{(u)}, 0\right) \neq 0$, and $\deg\left(id - Q_n, W_0, 0\right) \neq 0$ if $W_n = W_0$. This implies existence of $y = (0, w) \in W_0$ satisfying $Q_n^{(u)}(y) = 0$ and also existence of $y \in W_0$ satisfying $Q_n(y) = y$ if $W_n = W_0$. It remains to use Assertion (a) to establish the inclusions $y_i \in Y_i$ for the appropriate i. □

4.2 Finalizing the proof

Denote $S_{i,j} = \left\{ y \in \overline{V}_i : g_{i,j}^{(u)}(y) \in \overline{V}_j^{(u)} \right\}$, and $Z_i = \left(\bigcup_{a_{i,j}=1} \overline{S}_{i,j} \right) \bigcap \left(\bigcup_{a_{j,i}=1} g_{j,i}(\overline{S}_{i,j}) \right)$. By definition Z_i are compact, and they satisfy $Z_i \subset V_i$ by (2.1).

Lemma 4.2.

(α) *Let $\omega \in \Omega_A$ be a given p-periodic symbolic sequence. Then there exists a p-periodic trajectory $\mathbf{x} \in \mathrm{Tr}(f)$ satisfying $x_i \in h_i\left(Z_{\omega_i}\right)$. If $\gamma(\omega) < 0$ then the trajectory $\mathbf{x}$ can be chosen not asymptotically stable.*

(β) *Let $\omega \in \Omega_A$ be a given symbolic sequence. Then there exists a trajectory $\mathbf{x} \in \mathrm{Tr}(f)$ satisfying $x_i \in h_i\left(Z_{\omega_i}\right)$.*

P r o o f. (α) Let us consider the sequence of product-sets $W_i = V_{\omega_i}, \quad i = 0, 1, \ldots, p$ and the sequence of mappings $g_i = h_{\omega_i}^{-1} f h_{\omega_{i-1}}$, $i = 1, \ldots, p$. By the second assertion of Corollary 4.1 there exists a 'g-trajectory' $\mathbf{y}$ satisfying $y_i \in Y_i$, $i = 0, \ldots, p-1$, and $y_p = y_0$. Then the p-periodic sequence $\mathbf{x}$ which is defined by $x_i = h_{\omega_i}(y_i)$ an f-trajectory satisfying $x_i \in h_{\omega_i}\left(Y_i\right) \subseteq h_{\omega_i}\left(Z_{\omega_i}\right)$.

It remains to consider the case when $\gamma(\omega) < 0$. In this case

$$\deg\left(id - Q_p, W_0, 0\right) = (-1)^{d_u} \prod_{i=1}^{n} \deg(g_i^{(u)}(\cdot, 0), W_{i-1}^{(u)}, 0) = \gamma(\omega) < 0,$$

by the assertion (c), Lemma 4.1. By assertion (a), Lemma 4.1, $y_i = Q_i(y_0) \in S_i$. for any fixed point y_0 of Q_p, and y_0 must be a p-periodic for f by (4.4). Thus, if not all fixed points of Q_p are isolated, then we have nothing to prove. Otherwise, by the Kronecker formula, there exists a fixed point $y_0 \in S_0$ with a negative Kronecker index: $\mathrm{ind}(y_0, id - Q_p) < 0$. Again, by assertion (a), Lemma 4.1, $y_i = Q_i(y_0) \in S_i$, and by (4.4) $\mathrm{ind}(y_0, id - Q_p) = \mathrm{ind}(y_0, id - G_p) < 0$. Since h_i are homeomorphisms, $\mathrm{ind}(h_{\omega_0} y_0, id - f^p) = \mathrm{ind}(y_0, id - G_p) < 0$. Therefore, $x_0 = h_{\omega_0}(y_0)$ is a periodic point of f with the minimal period p and with a negative Kronecker index. This point can not be asymptotically stable, for instance, by Theorem 31.1, [11].

(β) Analogously, making use of the first assertion in Corollary 4.1, we can establish that for any $\omega \in \Omega_A$ and any positive integer n there exists a trajectory $x_{-n}, x_{1-n}, \ldots, x_n$ satisfying $x_i \in h_{\omega_i}\left(Z_{\omega_i}\right)$. Taken coordinate-wise limit we conclude that for any given sequence $\omega \in \Omega_A$ there exists a trajectory satisfying $x_i \in h_{\omega_i}\left(Z_i\right)$ for all integer i. □

Introduce a multi-valued operator Ψ which corresponds to each $\omega \in \Omega_A$ the set of trajectories $\mathbf{x}$ satisfying $x_i \in X_i$, with the additions that $\mathbf{x}$ must be p-periodic for a p-periodic ω, and $\mathbf{x}$ is not asymptotically stable if $\gamma(\omega) < 0$. The set $\Psi(\omega)$ is nonempty by the lemma above. To complete the proof of the theorem, it remains to apply the Zorn's Lemma to construct a single-valued selector of Ψ which is shift invariant. Indeed, let us denote by Φ the totality of single-valued functions ψ which are defined on subsets of $\mathcal{D}(\psi) \subset \Omega_A$ and satisfy the properties (p1)–(p3), and consider this set as being partially ordered by inclusion of the corresponding graphs $\mathrm{Gr}(\psi) = \{(\omega, \psi(\omega)) : \omega \in \mathcal{D}(\psi)\}$. By the construction every chain $\widehat{\Phi}$ (that is, linearly ordered subset) of Φ has an upper bound, the graph of which is defined as the union $\bigcup_{\psi \in \widehat{\Phi}} \mathrm{Gr}(\psi)$.

By Zorn's lemma there exists a maximal element φ in the set Φ. Suppose that the strict inclusion $\mathcal{D}(\varphi) \subset \Omega_A$ holds. Then there exists an element $\omega^* \in \Omega_A \setminus \mathcal{D}(\varphi)$. If for some positive integer i the sequence ω^* is the ith-shift of a sequence $\omega^0 \in \mathcal{D}(\varphi)$ then the mapping

$$\varphi_0(\omega) = \begin{cases} \varphi(\omega) & \text{if } \omega \in \mathcal{D}(\varphi), \\ \sigma_f^{-i} \varphi(\omega^0) & \text{if } \omega = \omega^* \end{cases}$$

satisfies the properties (p1)–(p3) and strictly dominates φ, which contradicts the definition of φ. On the other hand, if the sequence ω^* cannot be represented as a shift of a sequence $\omega \in \mathcal{D}(\varphi)$ then define $\varphi_0(\omega^*)$ as an arbitrary element from the nonempty set $\Psi(\omega^*)$. The mapping φ_0 again satisfies (p1)–(p3) and strictly dominates φ, and we arrive again at a contradiction.

The theorem and the lemma are proved. □

5 Examples

5.1 Example 1

Consider a Henon mapping $H_{a,b}(x^{(1)}, x^{(2)}) = (1 + x^{(2)} - a{x^{(1)}}^2, bx)$ with $a = a_* = 1.3924$, $b = b_* = 0.3$. We are interested in the Henon mapping with these particular values of parameters by the following considerations. For some values of parameters a, b, satisfying $|a - a_*|, |b - b_*| < 0.00005$, a fixed point

$$x_{a,b} = \left((b-1)/2a - \sqrt{((b-1)/2a)^2 + 4}\right)(1, b)$$

of this mapping generates stable and unstable manifolds which are tangent at some point (i. e. there is a *homoclinic tangency*). By the classical results of Mora and Viana [13] this implies abundance of strange attractors for generic diffeomorphisms sufficiently close to

H_{a_*,b_*}. Note also, that the Henon mapping with the classical parameters was investigated by Zgliczyński's method in [21, 7].

We will use the eigenvectors $v_1 \approx (-0.988, 0.155)$, $v_2 \approx (-0.462, -0.887)$ corresponding to the approximate eigenvalues -1.929, 0.156 of the linearization of H_{a_*,b_*} at the fixed point

$$x_* = \left((b-1)/2a + \sqrt{((b-1)/2a)^2 + 4}\right)(1, b).$$

For a given $x = (x^{(1)}, x^{(2)}) \in \mathbb{R}^2$ we consider the homeomorphism $h_x : \mathbb{R}^1 \times \mathbb{R}^1 \mapsto \mathbb{R}^2$ as $h_x(y^{(u)}, y^{(s)}) = x + y^{(u)}v_1 + y^{(s)}v_2$. Let us construct an 11-element sequence x_i and corresponded rectangles

$$V_i = \left\{(y^{(u)}, y^{(s)}) : |y_i^{(u)}| < \alpha_i^{(u)}, |y_i^{(s)}| < \alpha_i^{(s)}\right\}$$

such that the mappings $g_{i,j} = h_{x_j}^{-1} H h_{x_i}$ be (V_i, V_j)-hyperbolic if $a_{i,j} = 1$, and A is the (11×11)-matrix (2.4).

Coordinates of the points $x_i = (x_i^{(1)}, x_i^{(2)})$, $i = 1, \ldots, 11$, and the sizes of the corresponded rectangles are presented in Table 1.

Table 1: Numerical values for $x_i^{(1)}$, $x_i^{(2)}$, $\alpha_i^{(u)}$ and $\alpha_i^{(s)}$

i	1	2	3	4	5	6	7	8	9	10	11
$x_i^{(1)}$	.632	.6045	.6854	.5273	.8184	.2256	1.1747	-.8537	.3376	.5852	.6245
$x_i^{(2)}$	.1897	.1941	.1813	.2056	.1582	.2455	.0677	.3524	-.2561	.1013	.1755
$\alpha_i^{(u)}$	.03	.025	.037	.05	.05	.05	.02	.027	.02	.016	.022
$\alpha_i^{(s)}$	.01	.01	.005	.004	.004	.004	.008	.0045	.01	.01	.01

Actually, we chose as x_i the elements of a 'quasi-homoclinic quasi-orbit' $\mathcal{O}$ which passes close to x_{a_*,b_*}; these elements were taken from [15], Table 3. The sizes of the rectangles were adjusted to satisfy the (2.1).

Lemma 5.1. *The family $\mathcal{U}$ of connected components of the set $U = \bigcup_{i=1}^{11} h_i(V_i)$ has eight components.*

P r o o f. See Figure 2 at the end of this section, or make necessary calculation with a pocket calculator. □

Lemma 5.2. *The mappings $g_{i,j} = h_{x_i}^{-1} H_{a_*,b_*} h_{x_j}$ are (V_i, V_j)-hyperbolic if $a_{i,j} = 1$ and the corresponding number (3.3) satisfies the estimate $\chi = \chi_* > 0.0001$. Moreover,* $\deg(g_{8,9}^{(u)}, V_i^{(u)}, 0) = 1$, *and* $\deg(g_{i,j}^{(u)}, V_i^{(u)}, 0) = -1$ *for other (i, j) satisfying $a_{i,j} = 1$.*

P r o o f. Consider Figures 4,5 at the end of the section. These pictures, combined with the last sentence in Section 2.2, prove the lemma for the 'unperturbed' mapping $f = H_{a_*,b_*}$. To estimate χ_* it remains to proceed with some additional trivial calculations. □

To make this 'picture proof' easily verifiable, and to encourage readers to prove other similar assertions, we included in the paper the scripts of Mathematica-IV programs which produced Figures 2–4 (see Appendix). We would like to stress once more that a computer is not necessary to prove this lemma: a pocket calculator would be more than enough. On the other hand, packages like Mathematica or Maple are recommended when locating the quasi-orbit x_i and 'online' adjusting sizes of rectangles.

Let now consider a mapping $H = (H_1, H_2) : \mathbb{R}^2 \times \mathbb{R}^{\widehat{d}} \to \mathbb{R}^2 \times \mathbb{R}^{\widehat{d}}$ that satisfies the inequalities

$$|H_1(x, y) - H_{a_*,b_*}(x)| \leq \varepsilon_1 + c_1|y|, \quad |H_2(x, y)| \leq \varepsilon_2 + c_2|y|.$$

Here $\varepsilon_1, \varepsilon_1, c_1, c_2 > 0$, $c_2 < 1$ and $\varepsilon_1 + c_2\varepsilon_2/(1 - c_2) < 0.0001 < \chi_*$.

Corollary 5.1. *Let A be the 11×11 matrix (2.4).*

(i) The mapping H is $(\mathcal{X}, \sigma_A)$-compatible, where X_i are compact subsets of $h_i(V_i)$.

(ii) H is $(\widehat{\mathcal{U}}, 20)$ chaotic, where $\widehat{\mathcal{U}}$ is the eight-element set, which projection to $\mathbb{R}^2$ consists of connected components of the set graphed in Figure 1.

(iii) For any $p \geq 11$ the function f has a periodic point $x \in X_1$ which has the minimal period p, and which is not asymptotically stable

P r o o f. Assertion (i) follows from Theorem 1. Assertion (ii) follows from Corollary 3.1. Assertion (iii) follows Lemma 3.2 and the last part of Lemma 5.2. □

5.2 Example 2

Due to Corollary 3.2 it is possible to estimate chaos threshold of the Hénon mapping (with the same parameters as above) with respect to an invariant set which is similar to the attractor. In this subsection we describe some numerical results in this direction.

Firstly we specify an invariant set. We construct the set $Z \subset \mathbb{R}^2$ by describing its boundary ∂Z. Let us define the points

$$p_0^0 = (0.55, 0.01),\ p_1^0 = (-1.12, -0.29),\ p_2^0 = (-1.33, 0.38),$$

$$p_0^1 = (1.29, -0.008),\ p_1^1 = (-1.1, -0.41),\ p_2^1 = (-1.33, 0.4)$$

and four functions (parabolas) $b_{i\in\{0,1\},j\in\{1,2\}}$:

$$x^{(2)} \in \mathbb{R} \mapsto\ x^{(1)} \in \mathbb{R}$$

$$b_{i\in\{0,1\},j\in\{1,2\}}(x^{(2)}) = \frac{(p_j^{i\,(1)} - p_0^{i\,(1)})}{(p_j^{i\,(2)} - p_0^{i\,(2)})^2}(x^{(2)} - p_0^{i\,(2)})^2 + p_0^{i\,(1)}$$

satisfying $b_{i\in\{0,1\},j\in\{1,2\}}(p_j^{i\,(2)}) = p_j^{i\,(1)}$ and $b_{i\in\{0,1\},j\in\{1,2\}}(p_0^{i\,(2)}) = p_0^{i\,(1)}$. Now, we connect $p_0^{i=0,1}$ to $p_{j=1,2}^i$ with function $b_{i,j}(x^{(2)})$ and $p_{j=1,2}^0$ to p_j^1 with straight lines. Z is the set bounded by this construction and it is invariant with respect to $H_{a,b}$. By Z^8 we denote ${H_{a,b}}^8(Z)$ (see Figure 5 at the end of this paper). This set is by definition also invariant for $H_{a,b}$, and it is the approximation from above for the attractor which we will use.

As the second step, using special computer algorithms[1], we will find large family of product sets as in the previous subsection. Because of numerical reasons (no problems with rounding in h_i^{-1}) it is convenient to redefine the vectors v_1, v_2 used in the homeomorphism,

$$h_x : \mathbb{R}^1 \times \mathbb{R}^1 \mapsto \mathbb{R}^2$$

$$h_x(y^{(u)}, y^{(s)}) = x + y^{(u)}v_1 + y^{(s)}v_2,$$

so we set $v_1 = (-1, 0)$, $\quad v_2 = (-0.5, -1)$. All trajectories $\mathbf{x} = \{x_i\}$ used in this example start from random points in the circle with the radius 0.01 centred at the fixed point

$$x_* = \left((b-1)/2a + \sqrt{((b-1)/2a)^2 + 4}\right)(1, b)\,.$$

and end in the same area — the last set V_i must contain the point x_*. The computer algorithm starts with the first rectangle (size as in 5) and tries to construct next one satisfying (2.1) and so on, until trajectory will be back and $x_* \in V_i$ will be valid (see Figure 6). If fails with one starting point, goes with another one. It was found 237669 such $\mathbf{x}$ and each has not less than 20 points. We choose 161 trajectories in which at least one element is further then 1.799 from x_* to make set $\{\mathbf{x} = \{x_i\}\}$ more dense in Z^8 and presentation more cleary. One trajectory, together with the sizes of corresponding rectangles, is printed in Appendix; the others can be found in the electronic version

[1] it will be described elsewhere

of this preprint in the home page of Institute for Nonlinear Sciences: www.ins.ucc.

Due to Corollary 3.2 we found for this example $r = 0.0341$ and $\Delta = 0.31$ (see Figure 7). The chaos threshold with respect to Z^8 estimated in this section is to be not greater than 0.345.

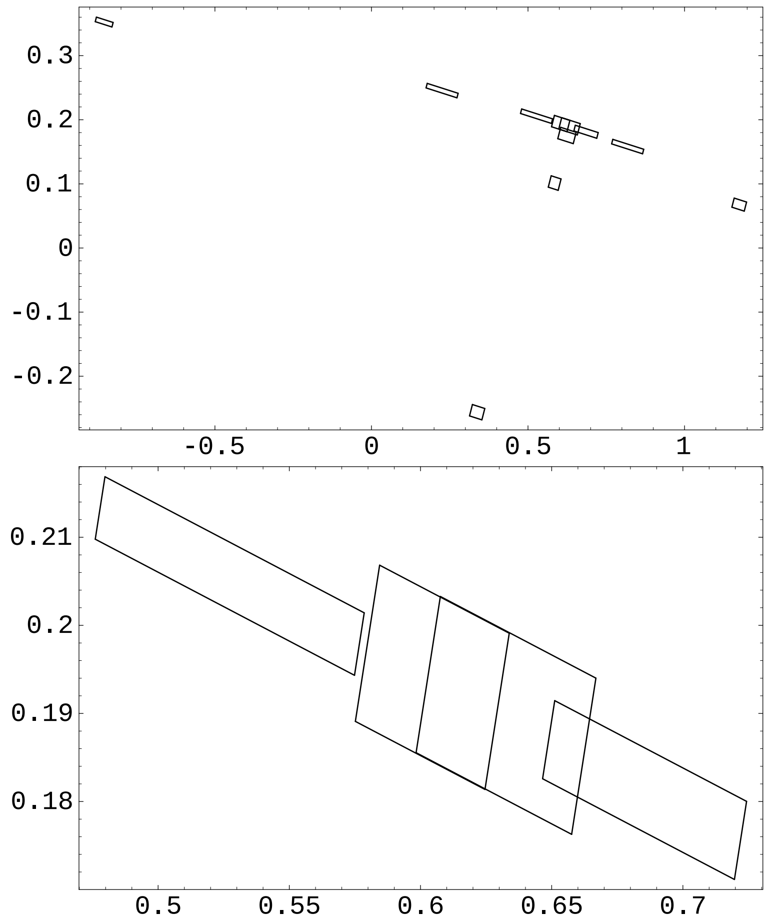

Figure 2: Above: the sets $h_i(V_i)$ for $i = 1, \ldots, 11$. Below: the sets $h_i(V_i)$ for $i = 1, \ldots, 4$.

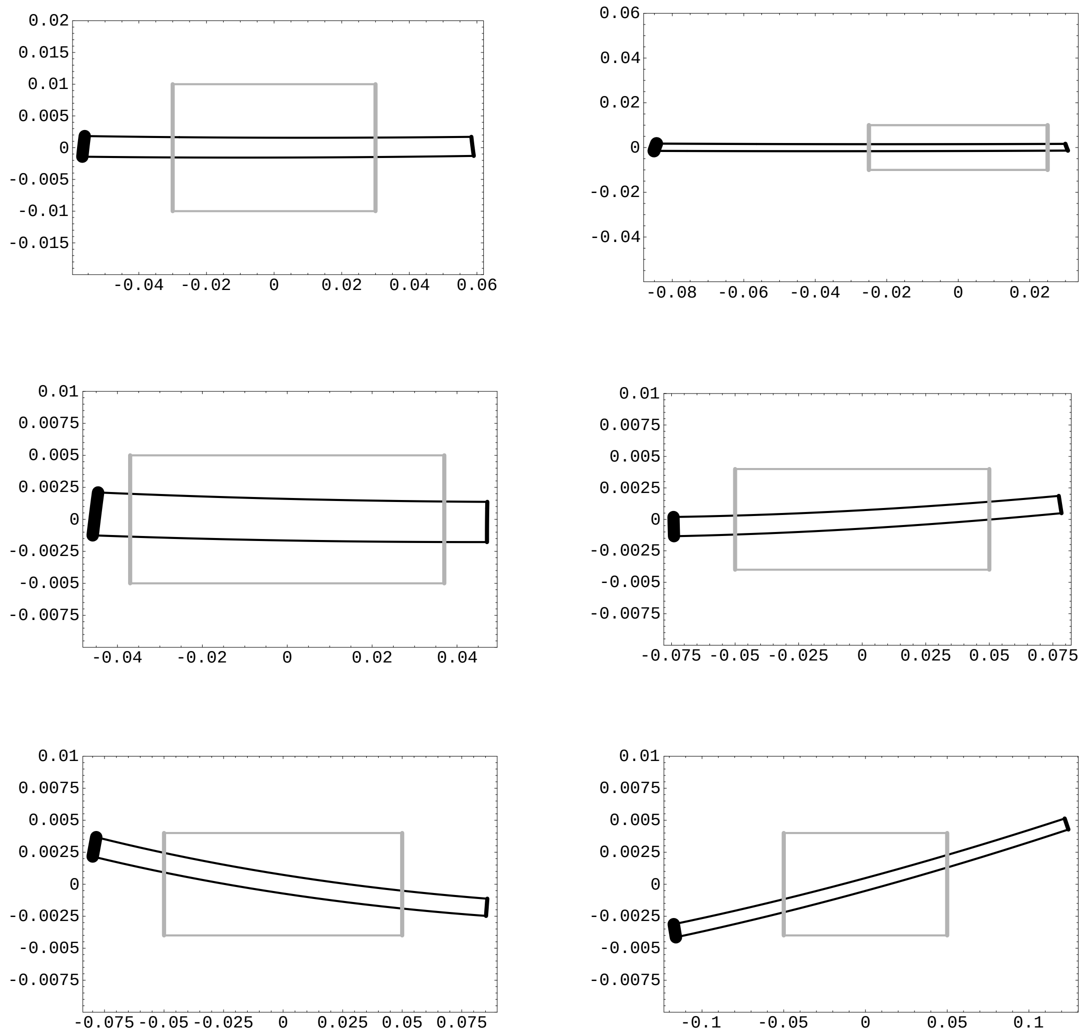

Figure 3: The first picture: $g_{1,1}(V_1)$ (black) against V_1 (gray); the other 5 pictures: and $g_{i,i+1}(V_i)$ against V_{i+1} for $i = 1, 2, 3, 4, 5$. The sets $g_{i,j}(\partial V_i^{(u)} \times V_i^{(s)}$ and $\partial V_j^{(u)} \times V_j^{(s)}$ are bolder than $g_{i,j}(V_i^{(u)} \times V_i^{(s)})$ and $V_j^{(u)} \times \partial V_j^{(s)}$. The $g_{i,j}$-images of the left part of $\partial V_i^{(u)} \times V_i^{(s)}$ are very bold.

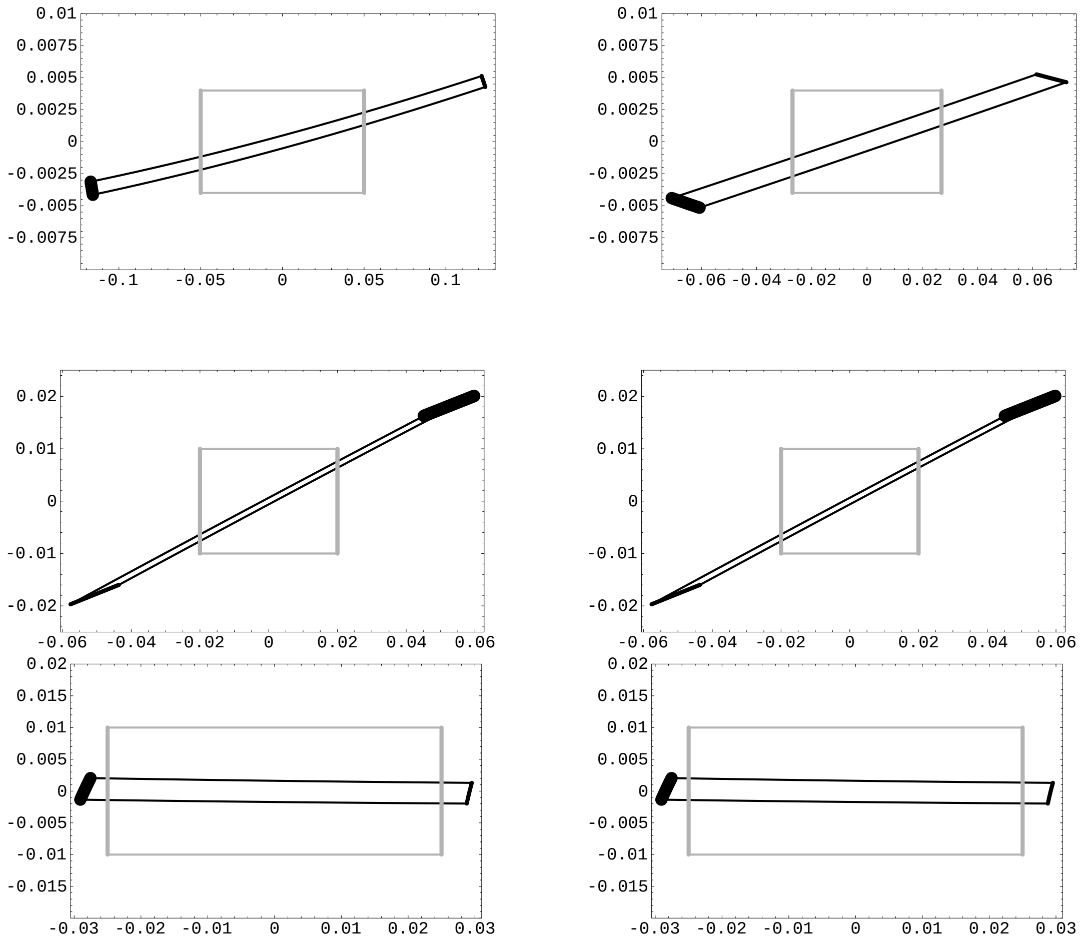

Figure 4: $g_{i,i+1}(V_i)$ against V_{i+1} for $i = 6, 7, 8, 9, 10$ and $g_{11,1}(V_{11})$ against V_1.

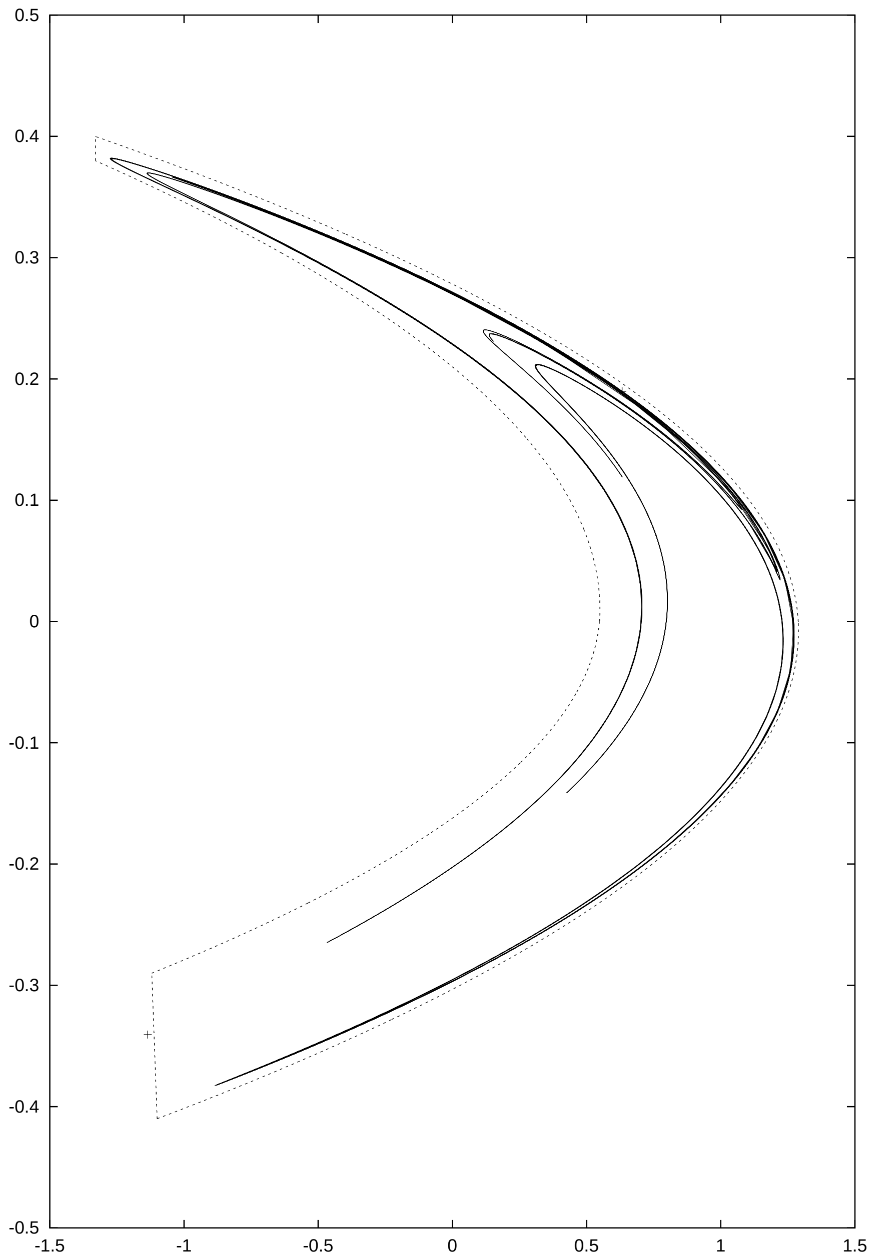

Figure 5: The set Z^8 (solid line) plotted against Z; two crosses denote fixed points.

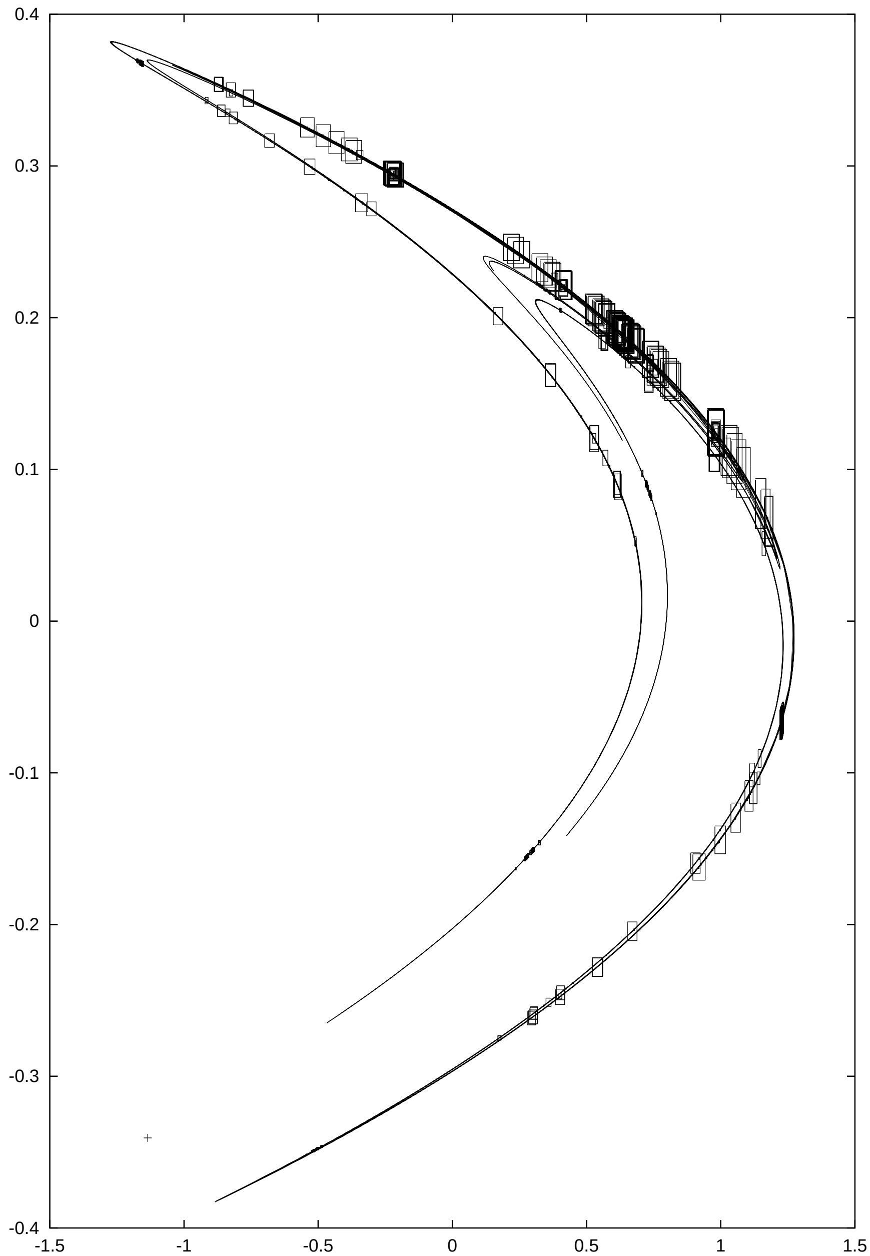

Figure 6: The set Z^8 (aproximation of Hénon set) plotted against the sets V_i (whole number 4758), two crosses denote fixed points.

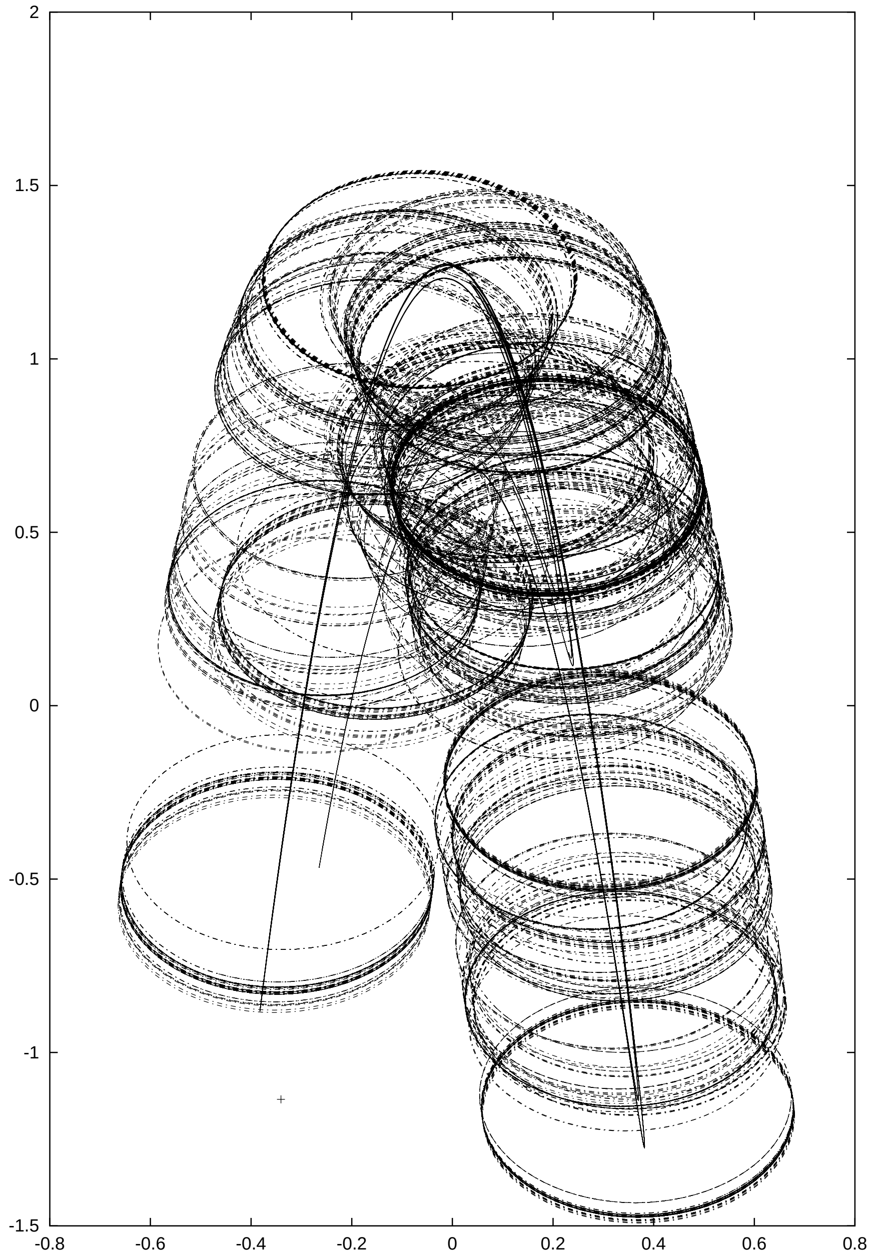

Figure 7: The inclusion $Z^8 \subset \bigcup B_{h_i(0)}(\Delta)$ for $\Delta = 0.31$. The crosses denote fixed points.

6 Appendix

6.1 Mathematica-IV programs to produce Figures 2 − 4

(* This script produces the top picture in Figure 2.
To produce the bottom picture, the line "pl[1],pl[2],pl[3],pl[4],pl[5], pl[6],pl[7],pl[8],pl[9],pl[10], pl[11],"" should be changed to "pl[1],pl[2],pl[3],pl[4],"*)

```
He[{x2_, y2_}] = {1 + y2 - a*x2*x2, 0.3*x2}; a = 1.3924; b = 0.3;
q = (1 - b)/(2a); fp = (Sqrt[q^2 + 1/a] - q){1, b}; mt = {{-2*a*fp[[1]], 1.}, {b, 0}};
ev = Eigenvectors[mt]; mt2 = Inverse[ev];
cp = {fp,fp,{0.6045,0.1941},{0.6854,0.1813},{0.5273,0.2056},{0.8184,0.1582},{0.2256,
     0.2455},{1.1747,0.06767},{-0.8537,0.3524},{0.3376,-0.2561},{0.5852,0.1013},
     {0.6325,0.1874},fp};
al = {{0.03,0.01},{0.03,0.01},{0.025,0.01},{0.037,0.005},{0.05,0.004},{0.05,0.004},
     {0.05,0.004},{0.02,0.008},{0.027,0.004},{0.02,0.01},{0.016,0.01},{0.025,0.01},
     {0.03,0.01}};
 For[i=1,i<12,i++,
co=cp[[i]];v1=al[[i]][[1]]*ev[[1]];v2=al[[i]][[2]]*ev[[2]];
pl[i]=ParametricPlot[{
{(v1+v2*t)[[1]]+co[[1]],(v1+v2*t)[[2]]+co[[2]]},{(v2+v1*t)[[1]]+co[[1]],(v2+v1*t)
[[2]]+co[[2]]},{(-v1+v2*t)[[1]]+co[[1]],(-v1+v2*t)[[2]]+co[[2]]},{(-v2+v1*t)[[1]]+
co[[1]],(-v2+v1*t)[[2]]+co[[2]]}},{t,-1,1}]];
Show[pl[1],pl[2],pl[3],pl[4],pl[5],pl[6],pl[7],pl[8],pl[9],pl[10],pl[11],
Axes->None,Frame->True,PlotRange->All]
```

(* The script below produces the last picture in Figure 4. To produce, for instance, the first picture in Figure 3, the command "n=12" below should be changed to "n=1". *)

```
He[{x2_, y2_}] = {1 + y2 - a*x2*x2, 0.3*x2}; a = 1.3924; b = 0.3;
q = (1 - b)/(2a); fp = (Sqrt[q^2 + 1/a] - q){1, b}; mt = {{-2*a*fp[[1]], 1.}, {b, 0}};
ev = Eigenvectors[mt]; mt2 = Inverse[ev];
cp = {fp,fp,{0.6045,0.1941},{0.6854,0.1813},{0.5273,0.2056},{0.8184,0.1582},{0.2256,
      0.2455},{1.1747,0.06767},{-0.8537,0.3524},{0.3376,-0.2561},{0.5852,0.1013},
     {0.6325, 0.1874},fp};
al = {{0.03,0.01},{0.03,0.01},{0.025,0.01},{0.037,0.005},{0.05,0.004},{0.05,0.004},
     {0.05,0.004},{0.02,0.008},{0.027,0.004},{0.02,0.01},{0.016,0.01},{0.025,0.01},
     {0.03,0.01}};
n = 12;
ao = al[[n]]; an = al[[n + 1]]; co = cp[[n]]; cn = cp[[n + 1]];
v1 = ao[[1]]*ev[[1]]; v2 = ao[[2]]*ev[[2]];
h1 = ParametricPlot[{
{((He[v1+v2*t+co]-cn).mt2)[[1]],((He[v1+v2*t+co]-cn).mt2)[[2]]},
{((He[-v1+v2*t+co]-cn).mt2)[[1]],((He[-v1+v2*t+co]-cn).mt2)[[2]]},
{((He[v2+v1*t+co]-cn).mt2)[[1]],((He[v2+v1*t+co]-cn).mt2)[[2]]},
{((He[-v2+v1*t+co]-cn).mt2)[[1]],((He[-v2+v1*t+co]-cn).mt2)[[2]]},
{an[[1]], an[[2]]*t}, {-an[[1]], an[[2]]*t},
{an[[1]]*t, an[[2]]}, {an[[1]]*t, -an[[2]]}},
{t, -1, 1},
PlotStyle->{{Thickness[0.03]},{Thickness[0.01]},{Thickness[ 0.005]},
           {Thickness[0.005]},  {Thickness[0.01],GrayLevel[0.7]},
   {Thickness[0.01], GrayLevel[0.7]}, {Thickness[0.005],
   GrayLevel[0.7]}, {Thickness[0.005], GrayLevel[0.7]}},
           Axes -> None, Frame -> True, PlotRange -> {-0.02, 0.02}]
```

6.2 Sample numerical values for Example 2

i	$x_i^{(1)}$	$x_i^{(2)}$	$\alpha_i^{(u)}$	$\alpha_i^{(s)}$
1	0.633233262735324766756524911205258	0.189782120425760776933831970635397	0.0300000	0.0100000
2	0.631451490551827621494388066149965	0.189969978820597430026957473361577	0.0299700	0.0103200
3	0.634776955417815591760747893496190	0.189435447165548286448316419844989	0.0299400	0.0103889
4	0.628379308336016552723054594516833	0.190433086625344677528224368048857	0.0299101	0.0103975
5	0.640629249641654755185693726245766	0.188513792500804965816916378355050	0.0298802	0.0103917
6	0.617064707155576377861709510518119	0.192188774892496426555708117873730	0.0298503	0.0103824
7	0.662006224230109479233331298898778	0.185119412146672913358512853155436	0.0298204	0.0110887
8	0.574896991890492011544864097665191	0.198601867269032843769999389669633	0.0297906	0.0105339
9	0.738404545260165827935546138656690	0.172469097567147603463459229299557	0.0297608	0.0118228
10	0.413275149792629034288565910854155	0.221521363578049748380663841597007	0.0297311	0.0092658
11	0.983704526623196307551100067345900	0.123982544937788710286569773256247	0.0297013	0.0150676
12	-0.223407562113452568794683517305720	0.295111357986958892265330020203770	0.0296716	0.0078932
13	1.225615366788644276489361973476075	-0.067022268634035770638405055191716	0.0033279	0.0100328
14	-1.158592295858339712781788288698079	0.367684610036593282946808592042822	0.0033246	0.0015021
15	-0.501384186773655261487851522635037	-0.347577688757501913834536486609424	0.0033213	0.0006147
16	0.302392301778021378195273644737929	-0.150415256032096578446355456790511	0.0031257	0.0014467
17	0.722262150515176525552003546579347	0.090717690533406413458582093421379	0.0023489	0.0018475
18	0.364354666706783636235429284899462	0.216678645154552957665601063973804	0.0023466	0.0009508
19	1.031831525599084701112991648346410	0.109306400012035090870628785469839	0.0023224	0.0013545
20	-0.373148876237280307881697133929807	0.309549457679725410333897494503923	0.0023201	0.0006038
21	1.115671564944884522379514269017835	-0.111944662871184092364509140178942	0.0011838	0.0007871
22	-0.845097024918096933697366500266341	0.334701469483465356713854280705350	0.0011827	0.0003026
23	0.340264731607472568159972055525215	-0.253529107475429080109209950079902	0.0011815	0.0003248
24	0.585258698583880322582954223850233	0.102079419482241770447991616657564	0.0011466	0.0006451
25	0.625143788363343119020301796298662	0.1755577609575164096774886267155070	0.0011454	0.0004850
26	0.631421067140765302160097327862777	0.187543136509002935706090538889598	0.0011443	0.0004188
27	0.632403610354768744287438238768069	0.189426320142229590648029198358833	0.0011431	0.0004026
28	0.632557764077147024471619289501441	0.189721083106430623286231471630421	0.0011420	0.0003984
29	0.632581011123635715759631309543990	0.189767329223144107341485786850432	0.0011408	0.0003971
30	0.632586305726082171053458703923064	0.189774303337090714727889392863197	0.0011397	0.0003965
31	0.632583952767910002880422411425322	0.189775891717824651316037611176919	0.0011386	0.0003960
32	0.632589686174037898353474883343348	0.189775185830373000864126723427596	0.0011374	0.0003956
33	0.632578880159106778839250367307550	0.189776905852211369506042465003004	0.0011363	0.0003952
34	0.632599636280659419963849770756660	0.189773664047732033651775110192265	0.0011351	0.0003948
35	0.632559829774990819479227529149699	0.189779890884197825989154931226998	0.0011340	0.0003944
36	0.632636180071908055810428917528656	0.189767948932497245843768258744910	0.0011329	0.0003940
37	0.632489734938284715300621003596721	0.189790854021572416743128675258597	0.0011317	0.0003936
38	0.632770612110847589711925319529616	0.189746920481485414590186301079016	0.0011306	0.0003933
39	0.632231843631282620078553179821938	0.189831183633254276913577595858885	0.0011295	0.0003929
40	0.633265087882450160131971750847665	0.189669553089384786023565953946581	0.0011284	0.0003925

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