Symplectic Geometry

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Content:

Symplectic geometry is a branch of differential geometry dealing with the rather rigid structure on even-dimensional manifolds that results from specifying a nondegenerate closed 2-form. It has deep roots in classical mechanics, and has played a fundamental rôle in the description of the process of "quantisation", which produces a quantum system starting from a classical one. More recently, it was realised that this same framework is extremely useful in the infinite-dimensional setting, and this has led to remarkable applications in low-dimensional gauge theory, moduli problems in algebraic geometry, integrable systems, construction of invariants of manifolds and many areas of topology.

This course will be a thorough introduction to the basic techniques of symplectic geometry starting from the very scratch. It will be accessible to students in Mathematics or Physics that have had an exposure to the rudiments of differential geometry. The focus will be in the finite-dimensional setting, but if time permits I will also illustrate the usefulness of the symplectic package in two-dimensional gauge theory — in particular, the study of the vortex equations on a compact Riemann surface.

The tentative syllabus is as follows:

- Introductory material: symplectic vector spaces, symplectic manifolds and the canonical symplectic structure; lagrangian submanifolds and symplectomorphisms.
- Local theory: Moser's trick; Darboux theorem; Weinstein's lagrangian neighbourhood theorem and lagrangian embeddings.
- Kähler geometry: almost complex structures, compatibility and integrability; Dolbeault operators and cohomology; Kähler manifolds and examples.
- Symplectic mechanics: hamiltonian vector fields; Legendre transform; geometric quantisation.
- Symplectic reduction: hamiltonian actions and moment maps; the Marsden-Weinstein construction; the Atiyah–Guillemin–Sternberg convexity theorem.
- Toric geometry: Delzant polytopes and the classification of symplectic toric manifolds; Duistermaat–Heckman localisation theorems.
- Applications to gauge theory: the Atiyah–Bott symplectic structure; the vortex equations; construction and geometry of moduli spaces via symplectic reduction.

Prerequisites:

I will assume that the students have got a thorough background in linear algebra, and have had an exposure to the most basic objects of differential geometry (differentiable manifolds, vector fields, differential forms, Lie derivatives, riemannian metrics, vector bundles, de Rham cohomology). Some elementary notions from mathematical analysis will also be needed (topologies, compactness, convergence, etc). Familiarity with Lie theory will be an advantage, but not essential.

Assessment:

This will be discussed when the course starts.

Bibliography:

The main reference for the lectures will be:

- A. Cannas da Silva: *Lectures on Symplectic Geometry*, 2nd edition, Springer Verlag, 2008

  This textbook is available online at:
  `http://www.math.ist.utl.pt/~acannas/Books/lsg.pdf`

Other useful references are:

- V.I. Arnol'd: *Mathematical Methods of Classical Mechanics*, 2nd edition, Springer Verlag, 1997
- S. Bates, A. Weinstein: *Lectures on the Geometry of Quantization*, American Mathematical Society, 1997
- D. MacDuff, D. Salamon: *Introduction to Symplectic Topology*, 2nd edition, Oxford University Press, 1999

References to specialised literature will be given in the lectures.