Symplectic Geometry

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Jagiellonian University, 2nd Semester 2009/10

Lecture summaries:

- **Lecture 1** (03.III.2010)
  Standard form theorem for skew-symmetric bilinear forms over $\mathbb{R}$. Symplectic vector spaces and canonical bases. Symplectic, isotropic, co-isotropic and lagrangian subspaces. Definition of symplectic manifold; first examples and properties. Symplectomorphisms between symplectic manifolds and the group of symplectomorphisms of a symplectic structure. Tautological 1-form and canonical symplectic 2-form on a cotangent space; the inclusion $\mathrm{Diff}(X) \subset \mathrm{Symp}(\mathrm{T}^*X)$ by lifting diffeomorphisms.

  Exercises: 1, 2, 3, 4, 5.

- **Lecture 2** (10.III.2010)
  Symplectic, isotropic, co-isotropic and lagrangian submanifolds of a symplectic manifold. Criterion for the image of a 1-form to be a lagrangian submanifold; generating functions. Conormal bundles to a submanifold as lagrangian submanifolds. The graph of a symplectomorphism as a lagrangian submanifold with respect to the twisted product symplectic structure. Construction of symplectomorphisms between cotangent bundles from generating functions of type (q, Q). Application to the description of geodesic flow.

  Exercises: 6, 7, 8, 9, 10.

- **Lecture 3** (17.III.2010)
  Isotopies and vector fields. Normal space to a submanifold. Tubular neighbourhoods and their associated fibrations. Local exactness of a closed form vanishing on a submanifold. The homotopy formula in de Rham theory. Strongly isotopic, deformation equivalent and isotopic symplectic structures. The Moser trick. Two versions of Moser theorem on cohomologous symplectic structures on a compact manifold.

  Exercises: 11, 12, 13, 14.

- **Lecture 4** (31.III.2010)
  Moser theorem relative to a compact submanifold. Darboux theorem and Darboux coordinates. Lagrangian complements and linear symplectomorphisms determined by a complement to a lagrangian subspace. Weinstein lagrangian neighbourhood theorem. Canonical isomorphism between the normal bundle of a lagrangian submanifold and its cotangent

bundle. Weinstein tubular neighbourhood theorem; classification of lagrangian embeddings of compact manifolds up to local symplectomorphism.

Exercises: 15, 16, 17, 18.

- **Lecture 5** (07.IV.2010)

  Complex structures on a real vector space. Existence of compatible complex structures on a symplectic vector space. Almost complex structures on a manifold. The compatible almost complex structure on a symplectic manifold determined by a riemannian structure. Compatible triples on manifolds. Path-connectedness of the space of symplectic structures compatible with an almost complex structure. Almost complex submanifolds as symplectic submanifolds with respect to a compatible symplectic structure.

  Exercises: 19, 20, 21, 22, 23, 24, 25.

- **Lecture 6** (21.IV.2010)

  The J-holomorphic and J-antiholomorphic tangent and cotangent bundles of an almost complex manifold. Dolbeault splitting of complex-valued forms and Dolbeault differentials. J-holomorphic functions and J-holomorphic curves in almost complex manifolds. Complex manifolds and their canonical almost complex structures. Nuijenhuis tensor of an almost complex structure; the Newlander–Nirenberg theorem. Dolbeault splitting and differentials in complex manifolds; the Dolbeault complex.

  Exercises: 26, 27, 28, 29, 30.

- **Lecture 7** (27.IV.2010)

  Dolbeault cohomology groups of a complex manifold; Dolbeault's theorem. Kähler structures. Strictly plurisubharmonic functions and Kähler potentials. Kähler structures induced on a complex submanifold of a Kähler manifold. First examples of Kähler manifolds: flat metrics on $\mathbb{C}^n$, the Fubini–Study structure on $\mathbb{CP}^n$, closed orientable surfaces, flat metrics on complex tori, products of Kähler manifolds. Hodge decomposition theorem in the cohomology of a compact Kähler manifold.

  Exercises: 31, 32, 33, 34, 35.

- **Lecture 8** (28.IV.2010)

  Basic notions of Hodge theory on a compact Riemannian manifold: L^2 inner products, Hodge $*$-operator, the Laplace–Beltrami operator Δ_d, harmonic forms and the Hodge theorem on de Rham cohomology, Hodge decomposition on forms. Sketch of Hodge theory for compact Kähler manifolds: Hodge identities, Δ_d preserves bidegree, Hodge theorem on Dolbeault cohomology. Hodge diamond of a compact Kähler manifold. Elementary topological properties of compact Kähler manifolds. Hamiltonian and symplectic vector fields. Hamilton's equations on a cotangent bundle. The Poisson bracket and its algebraic properties.

  Exercises: 36, 37, 38, 39.

- **Lecture 9** (05.V.2010)

  Integrals of motion in a hamiltonian system. Completely integrable systems. The Liouville–Arnol'd theorem, Liouville tori and action-angle variables. Typical features of phase portraits of hamiltonian systems. Lagrangian mechanics: action integral, variational principle and the Euler–Lagrange equations. Legendre transform and dual function for a strictly convex function on a vector space.

Exercises: 40, 41, 42, 43, 44, 45.

- **Lecture 10** (11.V.2010)

  Equivalence between Lagrangian and canonical Hamiltonian systems via Legendre transform. Motivations for geometric quantisation. Prequantisations of a symplectic manifold. Construction of prequantisation data on a simply connected symplectic manifold. Classification of prequantisation data via flat line bundles. Weil prequantisation conditions. The prequantum operator associated to a classical observable.

  Exercises: 46, 47, 48, 49.

- **Lecture 11** (12.V.2010)

  Polarisations of a symplectic manifold. Real and Kähler polarisations; examples. Polarised sections of a prequantisation and change of polarisation under the flow of a hamiltonian vector field. Blattner–Kostant–Sternberg integral operators. Holomorphic quantisation of a Kähler manifold. Brief overview of further issues in geometric quantisation. Symplectic actions of a Lie group. The adjoint and coadjoint actions. Moment and comoment maps; hamiltonian actions.

  Exercises: 50, 51, 52, 53, 54, 55.

- **Lecture 12** (26.V.2010)

  Lie algebra cohomology. Criteria for existence and uniqueness of moment maps for symplectic actions. Classification of moment maps by the annihilator of the commutator ideal. The Marsden–Weinstein construction of the symplectic quotient (at zero level) of a hamiltonian space of a compact Lie group. Symplectic reduction at other levels. Noether's theorem and Noether principle for hamiltonian spaces.

  Exercises: 56, 57, 58, 59, 60.

- **Lecture 13** (02.VI.2010)

  The Atiyah–Guillemin–Sterberg convexity theorem for hamiltonian torus actions. The moment polytope of a hamiltonian action on a compact symplectic manifold. Effective torus actions on symplectic manifolds. Definition of symplectic toric manifold and first examples. Delzant polytopes and Delzant theorem on the classification of symplectic toric manifolds.

  Exercises: 61, 62, 63, 64.

- **Lecture 14** (09.VI.2010)

  Reconstruction of a toric manifold from its Delzant polytope. Typical phenomena in symplectic localisation for toric hamiltonian spaces. The Duistermaat–Heckman measure on a Lie algebra induced by a moment map. The Radon–Nikodym derivative as a piecewise polynomial. Duistermaat–Heckman formula for a circle action on a compact symplectic manifold.

  Exercise: 65.

- **Lecture 15** (30.VI.2010)

  Review of G-connections on principal bundles and the covariant derivative on fibre bundles associated to them via a G-action. Generalities on gauge-theory equations and their moduli spaces. The Atiyah–Bott symplectic structure on the space of connections on a principal bundle over a closed surface. Curvature as a moment map for gauge transformations. Moduli spaces of flat connections as symplectic quotients. The vortex equations on fibre bundles with Kähler typical fibre over a Riemann surface. Vortices on line bundles over a compact Riemann surface: construction of the moduli spaces via symplectic reduction.

  Exercise: Read the article: O. García-Prada, *A direct existence proof for the vortex equations over a compact Riemann surface*, Bull. London Math. Soc. **26** (1994) 88–96.