Symplectic Geometry

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Exercises:

1. Consider the standard symplectic 2-form $\omega_0 = \sum_{i=1}^n \mathrm{d}x_i \wedge \mathrm{d}y_i$ on $\mathbb{R}^{2n}$. Describe the group of linear symplectomorphisms $\mathrm{Sp}_{2n}(\mathbb{R}) := \mathrm{GL}_{2n}(\mathbb{R}) \cap \mathrm{Symp}(\mathbb{R}^{2n}, \omega_0)$ explicitly as a matrix group and compute its dimension.

2. Show that any closed orientable smooth surface admits a symplectic structure.

3. Let (M, ω) be a symplectic manifold; show that M is necessarily orientable.

4. Which of the (unit) even-dimensional spheres $S^{2n} \subset \mathbb{R}^{2n+1}$, $n \in \mathbb{N}$, are symplectic?

5. Show that the set of oriented lines in $\mathbb{R}^3$ can be given a symplectic structure ω which admits $\mathrm{Diff}(S^2)$ as a subgroup of symplectomorphisms.

6. Let (M, ω) be a symplectic manifold and $\alpha \in \Omega^1(M, \mathbb{R})$ such that $\omega = -\mathrm{d}\alpha$.
   (a) Show that there is a unique vector field $v \in \mathcal{X}(M)$ for which $\iota_v \omega = -\alpha$.
   (b) If $g \in \mathrm{Symp}(M, \omega)$ preserves α (i.e. $g^*\alpha = \alpha$), prove that g commutes with all the elements of the one-parameter subgroup of $\mathrm{Diff}(M)$ generated by v.

7. Let X be a manifold, and consider $M = \mathrm{T}^*X$ with tautological 1-form α and canonical symplectic structure $\omega = -\mathrm{d}\alpha$. Show that any $g \in \mathrm{Symp}(M, \omega)$ preserving α must preserve the fibres of $\mathrm{T}^*X \to X$ and lie in the image of $\mathrm{Diff}(X) \hookrightarrow \mathrm{Symp}(M, \omega)$.

8. Let S be an isotropic submanifold of a symplectic manifold (M, ω). Show that S is lagrangian (i.e. also co-isotropic) if and only if $\dim S = \frac{1}{2} \dim M$.

9. Let X be a manifold and $M = \mathrm{T}^*X$ equipped with the canonical symplectic structure ω. Suppose that $f : X \times X \to \mathbb{R}$ is a generating function for some $\varphi \in \mathrm{Symp}(M, \omega)$.
   (a) Give a geometric interpretation for the fixed points of φ in terms of the function $f \circ i_\Delta : X \to \mathbb{R}$, where $i_\Delta : x \mapsto (x, x)$ is the diagonal inclusion.
   (b) Construct (locally) a generating function for the symplectomorphism $\varphi^{(2)} := \varphi \circ \varphi$.

10. Suppose that (X, g) is a Riemannian manifold which is geodesically convex and complete, and let $d : X \times X \to \mathbb{R}$ be the corresponding metric distance. The function $f(x, y) = -\frac{1}{2} d(x, y)^2$ generates a canonical transformation $\varphi \in \mathrm{Symp}(\mathrm{T}^*X, \omega)$, which can also be interpreted as a diffeomorphism $\tilde{\varphi}$ of $\mathrm{T}X$ (called *geodesic flow*) using the identification of $\mathrm{T}X$ with T^*X provided by the metric g. Show that $\tilde{\varphi}(x, v) = (\exp_x(v)(1), \frac{\mathrm{d}}{\mathrm{d}t} \exp_x(v)(t)|_{t=1})$ for $v \in \mathrm{T}_x X$.

11. Let $i : X \hookrightarrow \mathbb{R}^n$ be the inclusion of a submanifold into Euclidean space. Show that the normal bundle $\mathrm{N}X$ can be identified with the subbundle of $i^*\mathrm{T}\mathbb{R}^n \to X$ whose fibre at $x \in X$ is the orthogonal complement $(\mathrm{d}i|_x(\mathrm{T}_x X))^\perp \subset \mathrm{T}_{i(x)}\mathbb{R}^n$.

12. Give a complete proof of the tubular neighbourhood theorem for submanifolds.

13. Prove that two symplectic structures on a manifold M are isotopic if they are strongly isotopic (and thus symplectomorphic). Give a topological condition on M ensuring that the converse is true.

14. Let S be a closed surface and ω_0, ω_1 two symplectic structures on S with $\int_S \omega_0 = \int_S \omega_1$. Show that ω_0 and ω_1 are strongly isotopic.

15. Show that the following is an equivalent definition of symplectic structure on a $2n$-manifold M: an atlas $\{\phi_i, U_i\}_{i\in I}$ for M such that for $U_i \cap U_j \neq \emptyset$ one has $\phi_i \circ \phi_j^{-1} \in \mathrm{Symp}(\mathbb{R}^{2n}, \omega_0)$, where ω_0 is the standard symplectic 2-form.

16. Find global Darboux coordinates on a 2-sphere with two antipodal points removed, equipped with the symplectic structure induced from the usual (round) area 2-form.

17. Let (M, ω) be a compact symplectic manifold. Explain how the following statement can be made precise: the Lie algebra of the group $\mathrm{Symp}(M, \omega)$ is the vector space of closed 1-forms on M.

18. Recall that the C^1 topology on the set of diffeomorphisms of a manifold M is defined by the following notion of convergence: a sequence $(f_k)_{k\in\mathbb{N}}$ in $\mathrm{Diff}(M)$ is said to C^1-*converge* iff the sequence of derivatives $\mathrm{d}f_k : \mathrm{T}M \to \mathrm{T}M$ converges uniformly on compact sets. Now let (M, ω) be a compact symplectic manifold with $H^1(M; \mathbb{R}) = 0$. Prove that a symplectomorphism of M which is sufficiently C^1-close to the identity (meaning: contained a sufficiently small neighbourhood of id_M with respect to the C^1 topology) has at least two fixed points.

19. Let (V, Ω) be a symplectic vector space. Show that the set $\mathcal{J}(V, \Omega)$ of all Ω-compatible complex structures on V is contractible; in other words, there is a homotopy $h_t : \mathcal{J}(V, \Omega) \to \mathcal{J}(V, \Omega)$ (continuous with $0 < t < 1$) such that h_0 is the identity, h_1 is a constant element $J \in \mathcal{J}(V, \Omega)$ and $h_t(J) = J$ for all $t \in [0, 1]$.

20. Let (V, J) be a complex vector space. Show that there is a positive inner product $G(\cdot, \cdot)$ on V with respect to which the complex structure J is orthogonal. Use this fact to prove that V admits a symplectic structure Ω such that J is Ω-compatible.

21. Let $\Omega(V)$ and $J(V)$ denote the sets of symplectic and complex structures (respectively) on a vector space V, and fix $\Omega \in \Omega(V)$, $J \in J(V)$. Show that there are bijections $\Omega(V) \cong \mathrm{GL}(V)/\mathrm{Sp}(V, \Omega)$ and $J(V) \cong \mathrm{GL}(V)/\mathrm{GL}(V, J)$.

22. Given $n \in \mathbb{N}$, show that the intersection of any two of $\mathrm{O}(2n), \mathrm{Sp_{2n}}(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{C})$, identified as subgroups of $\mathrm{GL}_{2n}(\mathbb{R})$, is the unitary group $\mathrm{U}(n)$.

23. A symplectic structure on a (real) vector bundle $E \to M$ is a smooth section ω of $\Lambda^2(E^*) \to M$ for which the bilinear maps $\omega_p : E_p \times E_p \to \mathbb{R}$ are nondegenerate for all $p \in M$. Show that:

(a) If a vector bundle admits a symplectic structure, its structure group can be reduced from the linear group to a linear symplectic group, and conversely an $\mathrm{Sp}_{2n}(\mathbb{R})$-vector bundle of rank $2n$ admits a symplectic structure.

(b) A real vector bundle admits a symplectic structure if and only if it is also a complex vector bundle.

24. Let (V, Ω) be a symplectic vector space and J an Ω-compatible complex structure. Prove that L is a lagrangian subspace if and only if $J(L)$ is lagrangian, and that these two subspaces are orthogonal with respect to the inner product defined by the Ω-compatibility condition.

25. Show that the (unit) 6-dimensional sphere $S^6 \subset \mathbb{R}^7$ is an almost complex manifold.

26. Let (M, J) be an almost complex manifold, and for $v, w \in \mathcal{X}(M)$ set $N_J(v, w) := [Jv, Jw] - [Jv, w] - [v, Jw] - [v, w]$. Show that this defines a tensor $N_J \in \Gamma(M, \mathrm{T}M \otimes \mathrm{T}^*M^{\otimes 2})$, called the *Nijenhuis tensor* of (M, J).

27. Let $J \in \mathrm{End}(\mathbb{R}^{2n})$ be a complex structure, and suppose that there are n J-holomorphic functions $f_j : \mathbb{R}^{2n} \to \mathbb{C}$, $j = 1, \ldots, n$, such that the real and imaginary parts of all the $\mathrm{d}f_j$ at $p \in \mathbb{R}^{2n}$ form a basis of $\mathrm{T}_p\mathbb{R}^{2n}$. Show that $N_J|_p = 0$. Deduce that the Nijenhuis tensor of the canonical almost complex structure of a complex manifold vanishes everywhere.

28. Use the Newlander–Nirenberg theorem to show that any orientable closed surface is a smooth complex curve.

29. For $n \in \mathbb{N}$, the complex projective space $\mathbb{CP}^n$ is defined as the set of 1-dimensional subspaces of $\mathbb{C}^{n+1}$; in other words, it is the quotient $(\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^*$. Let $[z_0, z_1, \ldots, z_n]$ denote a point in this quotient, and for each integer $0 \leq j \leq n$ consider the map defined by $\varphi_j([z_0, \ldots, z_n]) := \left(\frac{z_0}{z_j}, \ldots, \frac{z_n}{z_j}\right)$ for points where $z_j \neq 0$, where the quotient $\frac{z_j}{z_j}$ has been omitted in the right-hand side. Show that these $n+1$ maps φ_j define an atlas giving $\mathbb{CP}^n$ the structure of a complex manifold.

30. Let J denote the canonical complex structure on $\mathbb{C}$, and consider the subgroup $\Lambda_\tau = \{k + m\tau \in \mathbb{C} : k, m \in \mathbb{Z}\} \cong \mathbb{Z}^2$ of $(\mathbb{C}, +)$ that is to each $\tau \in \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. Show that J descends to an integrable almost complex structure J_τ on the quotient $\mathbb{T}^2_\tau := \mathbb{C}/\Lambda_\tau \cong S^1 \times S^1$. When are two such $\mathbb{T}^2_\tau$ isomorphic as complex manifolds?

31. Let ω_0 and ω_1 be two cohomologous Kähler structures on a complex manifold M; prove that there exists a symplectomorphism between (M, ω_0) and (M, ω_1).

32. Calculate a global Kähler potential for the hyperbolic plane, which is the surface $\{x + iy \in \mathbb{C} : y > 0\}$ equipped with the Riemannian metric $\frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}$.

33. Show that $\mathbb{CP}^1$ with the Kähler metric defined by the Fubini–Study structure is isometric to a 2-sphere embedded in $\mathbb{R}^3$. What is its radius?

34. Show that $\mathrm{U}(n+1)$ acts transitively on $\mathbb{CP}^n$ by isometries of the Fubini–Study metric.

35. Let (M, J, ω) be a Kähler structure, and consider the Riemann curvature tensor R_∇ for the Levi-Civita connection ∇ of the underlying Kähler metric, defined by $R_\nabla(X, Y)Z := [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$ for $X, Y, Z \in \mathcal{X}(M)$. Recall that the Ricci tensor is defined by $\mathrm{Ric}(X, Y) := \mathrm{Tr}(Z \mapsto R_\nabla(Z, X)Y)$. Show that:

(a) The equation $\rho(X, Y) := \mathrm{Ric}(JX, Y)$ defines a closed 2-form ρ on M.

(b) The cohomology class $[\rho] \in H^2(M; \mathbb{R})$ does not depend on the choice of symplectic structure ω.

36. Let ω be a Kähler structure on a compact complex manifold M, and denote by Λ the L^2-adjoint of the operator $\alpha \mapsto \alpha \wedge \omega$ on forms. Use the Hodge identities $[\Lambda, \bar{\partial}] = -i\partial^*$ and $[\Lambda, \partial] = i\bar{\partial}^*$ to show that the Laplacians on M satisfy $\Delta_d = 2\Delta_\partial = 2\Delta_{\bar{\partial}}$.

37. Show that the complex surface $Q = \mathbb{CP}^1 \times \mathbb{CP}^1$ embeds in $\mathbb{CP}^3$ as a quadric hypersurface. Relate the Kähler structure on Q induced by the Fubini–Study structure $\omega_{\mathbb{CP}^3}$ to the product Kähler structure $\omega_1 + \omega_2$, where $\omega_j = \mathrm{pr}_j^* \omega_{\mathbb{CP}^1}$ for $j = 1, 2$ are the pull-backs of the Fubini–Study structure by the projections $\mathrm{pr}_j : Q \to \mathbb{CP}^1$ onto each factor. Construct the Hodge diamond of Q.

38. Let X be a vector field on a compact manifold Q and ρ_t^X its flow. Show that there is a unique vector field $X_\sharp$ on T^*Q whose flow satisfies $\pi \circ \rho_t^{X_\sharp} = \rho_t^X \circ \pi$ for $t \in \mathbb{R}$, where $\pi : \mathrm{T}^*Q \to Q$ is the standard projection, and that $X_\sharp$ is hamiltonian with respect to the canonical symplectic structure.

39. Show that the vector space of (real) smooth functions on a symplectic manifold (M, ω) is a Poisson algebra, when supplemented by the operations of pointwise multiplication and the Poisson bracket defined by ω.

40. Let (M, ω, H) be an integrable system with $\dim M = 2n$ and $f_1 = H, f_2, \ldots, f_n$ independent integrals of motion in involution; consider a connected component M_c of the level set $f^{-1}(\{c\})$ of the map $f = (f_1, \ldots, f_n) : M \to \mathbb{R}^n$, where $c \in \mathbb{R}^n$ is a regular value of f.

(a) If the fluxes $\rho_{t_i}^{X_{f_i}}$ of the hamiltonian vector fields associated to the f_i are defined for all $t_i \in \mathbb{R}$, show that the map $\mathbb{R}^n \times M_c \to M_c$ given by $((t_1, \ldots, t_n), p) \mapsto \phi_{t_n}^{X_{f_n}} \circ \cdots \circ \phi_{t_1}^{X_{f_1}}(p)$ defines a transitive action of the group $(\mathbb{R}^n, +)$ on M_c.

(b) Show that the stabilisator subgroup $\mathrm{Stab}_p\mathbb{R}^n$ of any $p \in M_c$ is a discrete subgroup of $\mathbb{R}^n$, hence a lattice $\Lambda \cong \mathbb{Z}^k$ of rank $k \leq n$.

41. Let $M_c \hookrightarrow M \cong \mathbb{T}^k \times \mathbb{R}^{n-k}$ be a submanifold as in exercise 40, and suppose that there is an open set $\mathcal{U} \subset M$ with $M_c \cap \mathcal{U} \neq \emptyset$, and $\alpha \in \Omega^1(\mathcal{U})$ such that $\omega|_{\mathcal{U}} = -\mathrm{d}\alpha$. Consider a noncontractible loop γ in $M_c \cap \mathcal{U}$. Show that the integral $\oint_\gamma \alpha$ does not change if one adds a closed 1-form to α, or deforms γ within the same homology class.

42. Consider the harmonic oscillator, which is the hamiltonian system on $\mathrm{T}^*\mathbb{R}$ (with canonical symplectic structure $\omega = \mathrm{d}x \wedge \mathrm{d}p$) defined by the hamiltonian $H(x, p) = \frac{1}{2m}p^2 + \frac{k}{2}x^2$, where m, k are positive constants. Calculate action-angle variables for $(\mathrm{T}^*\mathbb{R}, \omega, H)$ outside the critical point $(0, 0)$.

43. Prove Liouville's theorem: if (M, ω) is a symplectic manifold of dimension $2n$, the *symplectic volume* form $\frac{1}{n!}\omega^{\wedge n}$ is preserved by the flux of a hamiltonian vector field.

44. A Riemannian metric g on a manifold Q yields a function $T : \mathrm{T}Q \to \mathbb{R}$ by setting $T(q, \dot{q}) := \frac{1}{2}g_q(\dot{q}, \dot{q})$, where $q \in Q$ and $\dot{q} \in \mathrm{T}_qQ$. Consider the lagrangian system on Q defined by the

lagrangian T. Write down the corresponding Euler–Lagrange equations in local coordinates and interpret the motions geometrically.

45. Suppose $F : V \to \mathbb{R}$ is a strictly convex function on a vector space V with quadratic growth at infinity, i.e. $F(p) > Q(p)\ \forall_{p\in V}$ holds for some positive-definite quadratic form Q on V; prove that the stability set of F is the whole V^*. Show that the dual function $F^* : V^* \to \mathbb{R}$ also has maximal stability set $S_{F^*} = V$ and that the two Legendre transforms $\mathcal{L}_F : V \to V^*$ and $\mathcal{L}_{F^*} : V^* \to V$ are inverses of each other.

46. Let $0 < \phi < 2\pi$ and $0 < \theta < \pi$ denote polar coordinates on $S^2 \subset \mathbb{R}^3$, inducing coordinates p_ϕ, p_θ on the fibres of $\mathrm{T}^*S^2 \to S^2$, and $\ell, m, g > 0$. Show that $H : \mathrm{T}^*S^2 \to \mathbb{R}$ given by $(\phi, \theta, p_\phi, p_\theta) \mapsto \frac{\ell^2}{2m}(\frac{p_\phi^2}{\sin^2\theta} + p_\theta^2) + m\ell g\cos\theta$ can be interpreted as the energy of a spherical pendulum. Write down Hamilton's equations for H and verify that they are equivalent to the Euler–Lagrange equations for the dual function $H^* : \mathrm{T}S^2 \to \mathbb{R}$ obtained by fibrewise Legendre transform. Discuss the integrability of this system.

47. Consider the lattice $\Lambda_\tau := \{k + m\tau \in \mathbb{C} : k, m \in \mathbb{Z}\} \cong \mathbb{Z}^2$ where $\mathrm{Im}(\tau) > 0$, and let ω_τ be the Kähler structure on $\mathbb{T}^2_\tau := \mathbb{C}/\Lambda_\tau$ induced by the standard Kähler structure on $\mathbb{C}$. Find the values of τ and $\hbar$ for which $(\mathbb{T}_\tau, \omega_\tau)$ admits a prequantisation.

48. Consider the symplectic manifold $M = \mathrm{T}^*S^1$ with canonical symplectic structure $\omega = -\mathrm{d}\alpha$, where $\alpha = p\,\mathrm{d}\theta$ is the tautological 1-form. For $\hbar > 0$ and $\nu \in [0, 1[$, construct a prequantisation of (M, ω) using the operator $\nabla^{\hbar,\nu} := \mathrm{d} + i\hbar\,\alpha + i\nu\,\mathrm{d}\theta$, acting on smooth functions $f : M \to \mathbb{C}$. Determine the spectrum of the prequantum operator associated to the fibre coordinate p for each pair $(\hbar, \nu)$, and show that prequantisations corresponding to different pairs are inequivalent.

49. Let M be a manifold, $\tilde{M}$ its universal cover, and $\tilde{\alpha}$ the standard action of $\pi_1(M)$ on $\tilde{M}$. If $\rho : \pi_1(M) \to \mathrm{U}(1)$ is a group homomorphism, then $\pi_1(M)$ also acts on $\tilde{M} \times \mathbb{C}$ via $\alpha_{[\gamma]}(\tilde{p}, z) = (\tilde{\rho}_{[\gamma]}\tilde{p}, \rho([\gamma])z)$ for $[\gamma] \in \pi_1(M)$, $\tilde{p} \in \tilde{M}$ and $z \in \mathbb{C}$. Show that the space of orbits of this action is a complex line bundle over M, equipped with a flat connection, and that all flat line bundles on M arise in this way. Show that $b_1 = \dim H^1(M, \mathbb{R})$ determines the topology of the space of flat line bundles on M.

50. Let $i : S^2 \hookrightarrow \mathbb{R}^3$ be the standard unit sphere and ω the area form of the round metric $i^*(\sum_{j=1}^3 \mathrm{d}x_j{}^2)$ on S^2. For which $\ell, \hbar > 0$ can one construct a (unique) prequantisation of the Kähler manifold $(S^2, \ell\omega)$? Given such ℓ and $\hbar$, describe the action of the quantum operators associated to $i^*x_j : S^2 \to \mathbb{R}$ in holomorphic quantisation and compute the dimension of the quantum Hilbert space.

51. Let K be a strictly plurisubharmonic function on a bounded, connected and simply connected domain $\Omega \in \mathbb{C}^n$, and let $\omega = i\partial\bar{\partial}K$ be the associated Kähler 2-form. If $z = (z_1, \ldots z_n)$ are complex coordinates on $\mathbb{C}^n$, show that the quantum Hilbert space of (Ω, ω) in holomorphic quantisation can be described as the vector space $\mathcal{O}(\Omega)$ of holomorphic functions on Ω, with inner product given by $\langle\psi_1, \psi_2\rangle = \int_\Omega \overline{\psi_1(z)}\psi_2(z)e^{-\frac{1}{\hbar}K(z)}\frac{\omega^n}{n!}$.

52. Consider the hamiltonian action of a group G on a symplectic manifold (M, ω). Verify that the equivariance property of the moment map $\mu : M \to \mathfrak{g}^*$ is equivalent to the co-moment

map $\mu^* : \mathfrak{g} \to \mathcal{C}^\infty(M)$ being a Lie algebra homomorphism between $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathcal{C}^\infty(M)$ (equipped with the Poison bracket of ω).

53. Describe the orbits of the (adjoint) action of $\mathrm{U}(2)$ on the space $\mathfrak{u}(2)$ of 2×2 skew-hermitian matrices by conjugation.

54. Consider the Lie algebra $\mathfrak{g}$. If $f, g \in \mathcal{C}^\infty(\mathfrak{g}^*)$, let $\{f, g\}(\ell) := \langle \ell, [(\mathrm{d}f)_\ell, (\mathrm{d}g)_\ell] \rangle$ for each $\ell \in \mathfrak{g}^*$. Show that this operation turns $\mathcal{C}^\infty(\mathfrak{g}^*)$ into a Poisson algebra.

55. Show that, for $\eta \in \mathfrak{g}$, $X_\xi|_\eta = [\xi, \eta]$ coincides with the vector field generated by $\xi \in \mathfrak{g}$ via the adjoint representation of a Lie group G on $\mathfrak{g} = \mathrm{Lie}(G)$. Now for each $\ell \in \mathfrak{g}^*$ consider the skew bilinear form on $\mathfrak{g}$ given by $\omega_\ell(\xi, \eta) := \langle \ell, [\xi, \eta] \rangle$. Show that this restricts to a symplectic structure on each coadjoint orbit of G.

56. Consider the standard action of $\mathrm{SO}(3)$ on $\mathbb{R}^3$. Show that it lifts to a symplectic action on $\mathrm{T}^*\mathbb{R}^3 \cong \mathbb{R}^3 \times \mathbb{R}^3$ equipped with the canonical symplectic form. Show that the function $\mu(\vec{x}, \vec{p}) := \vec{x} \times \vec{p}$ can be interpreted as a moment map for this action.

57. If two hamiltonian actions of a Lie group G are given on the symplectic manifolds (M_1, ω_1) and (M_2, ω_2), show that the diagonal action of G on $(M_1 \times M_2, \mathrm{pr}_1^*\omega_1 + \mathrm{pr}_2^*\omega_2)$ is also hamiltonian.

58. Let G be a compact Lie group with a free action on a manifold M. Show that the space of orbits M/G is also a manifold, and that the projection $M \to M/G$ can be interpreted as a principal G-bundle.

59. Show that the action of $\mathrm{U}(r)$ on $\mathrm{Mat}_{r\times n}(\mathbb{C}) \cong \mathbb{C}^{r\times n}$ by left multiplication preserves the standard Kähler structure of $\mathbb{C}^{r\times n}$. Identifying $\mathfrak{u}(r)^*$ with $\mathfrak{u}(r)$ using the inner product $(A, B) \mapsto -\mathrm{tr}(\bar{A}^t B)$, show that all moment maps of this action are given by $\mu_\tau(W) = \frac{1}{2i}(W\bar{W}^t - \tau I_r)$, where I_r denotes the $r \times r$ unit matrix and $\tau \in \mathbb{R}$.

60. For $n > r$, show that the symplectic quotient corresponding to the moment map μ_1 in Exercise 59. at level 0 is the Grassmannian manifold $\mathrm{Gr}_r(\mathbb{C}^n)$ of r-dimensional subspaces of $\mathbb{C}^n$. Identify the symplectic structure on the quotient in the case $r = 1$.

61. Work out the details of the induction step in the proof (by induction on $m \in \mathbb{N}$) of connectedness of the level sets of any $\mathbb{T}^m$-moment map on a compact symplectic manifold (M, ω).

62. Let G be a compact Lie group and $H \subset G$ a closed subgroup; denote by $i^* : \mathfrak{g}^* \to \mathfrak{h}^*$ the projection dual to the inclusion of the Lie algebra of H into that of G. Show that from a hamiltonian space (M, ω, G, μ) one obtains another hamiltonian space $(M, \omega, H, i^*\mu)$ by restricting the G-action to H.

63. Let $\Delta \subset (\mathbb{R}^n)^*$ be a Delzant polytope. Show that the orientation of each of its facets is specified by a unique outward-pointing vector $v_i \in \mathbb{R}^n$ which is primitive in $\mathbb{Z}^n$.

64. Classify all Delzant polytopes in $\mathbb{R}^2$ with four vertices, up to translation, the action of $\mathrm{SL}_2(\mathbb{Z})$ and global rescaling.

65. Let $\Delta = [0, 1]^4 \subset \mathbb{R}^4$ be the unit hypercube. (An orthogonal projection of its 1-edges and vertices in two dimensions, known as a *tesseract*, is depicted on the webpage of this course.) Show that this is a Delzant polytope and describe the symplectic toric manifold $(M_\Delta, \omega_\Delta)$ associated to it via Delzant's theorem. What is the Euler characteristic of M_Δ?