# Mathematics and the ‘Dark Matter’ Puzzle

## based on Donald G. Saari, *N-body Solutions and Computing Galactic Masses* 2015 AJ 149 174

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## Abstract [Saari, 2015a]

A classical approach used to determine the mass distribution of a galaxy in terms of observed rotational velocities is applied to analytic solutions of Newtonian systems of $N$ discrete bodies (so mass distributions are known).

Predictions significantly exaggerate the amount of mass distributed at larger distances from the center; *e.g.*, rather than the actual 5% of mass on the outer edges, the method could predict over 80%.

Explanations are given for the differences.

## Central configuration

$N$ bodies define a *central configuration* (CC) if there is a common scalar $\lambda$ (depending on distances, masses, and other $N$-body variables) so that each body's position and acceleration vectors satisfy

$$\mathbf{r}_j'' = \lambda \mathbf{r}_j \ , \quad \text{which is} \quad \lambda \mathbf{r}_j = \frac{1}{m_j}\frac{\partial U}{\partial \mathbf{r}_j}, \tag{1}$$

$$U = \sum_{i<k} \frac{G m_i m_k}{r_{ik}}, \ j = 1, \ldots, N, \tag{2}$$

and $r_{ik} = |\mathbf{r}_i - \mathbf{r}_k|$ is the distance between particles.

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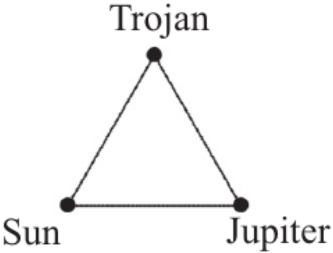


b. Lagrange

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- $n$ equal masses placed in an evenly spaced manner on a circle is a CC due to the symmetry;
- with additional body (of arbitrary mass) in the center of mass is also a $n+1$ body CC.

c. Maxwell

Moulton (1910) analyzed collinear central configurations.

### Moulton's collinear solutions

He proved that for any $N \geqslant 2$, any choice of masses $m_j$, and any ordering of the masses along a line, there exist unique (up to a common scalar multiple) spacings between adjacent particles that define a central configuration.

For any specified mass choices, there exist $N!/2$ collinear central configurations.

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For instance, specified mass choices define precisely four classes of three-body central configurations: three collinear choices (where the relative distances depend on the mass values) and the equilateral triangle (and its reflection).

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- To analyze the rings of Saturn, Maxwell placed his central configuration model into a circular orbit.
- In the next section, spiderweb central configurations are placed in circular orbits.

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Each circle has $2k$ masses (with two masses on each line), so this construction defines a $N = 2nk$ body configuration that resembles a spiderweb.

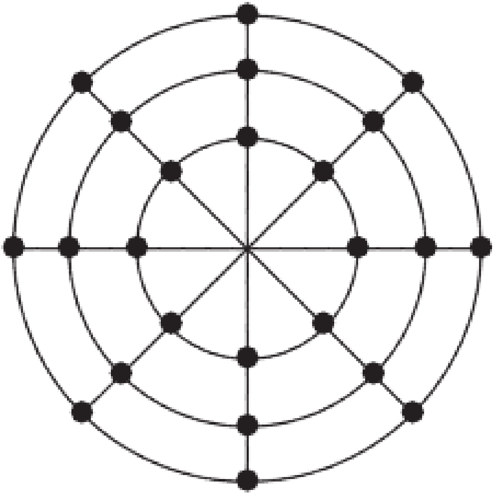

Figure: $k = 4$, $n = 3$ spiderweb central configuration.

**Theorem ([Saari, 2015a, Saari, 2015b])**

*For positive integers $k$ and $n$ and any choice of $m_j > 0$, $j = 1, \dots, n$, there exist unique spacings between the concentric circles so that the configuration is a spiderweb central configuration for these spacings and any positive multiple of them.*

## Idea of proof

- Symmetry requires particles on the $j$th circle to satisfy $\lambda_j \mathbf{r}_j = \frac{1}{m_j} \frac{\partial U}{\partial \mathbf{r}_j}$, so $\lambda_j$ is a smooth function of $m_j$'s and the positioning of particles.

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Warning: Saari's proof is incomplete!

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## 'Continuum' approach – assumptions

- based on Newton's first and second laws
- In a symmetric continuum setting, a body inside a spherical shell experiences no net gravitational force from the shell; the gravitational force on a body outside a spherical shell behaves as though the shell's matter is concentrated at the center.

- In the 2BP ($m$–$M$), the scalar acceleration of the mass $m$ satisfies ('centripetal and centrifugal force')

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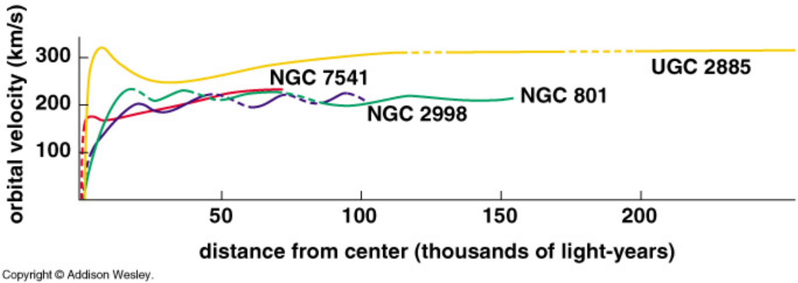
**'Continuum' approach [Binney and Tremaine, 2008]**

In symmetric settings where Newton's two laws apply, the mass up to radius $r$ from the center of mass, $M(r)$, has the form

$$M(r) = \frac{rv_{\mathrm{rot}}^2}{G} \tag{c}$$

where $v_{\mathrm{rot}}$ is the circular velocity of a star at distance $r$.




The observations of rotational velocities $v_{\text{rot}}$ of objects vs. their distance $r$ to galaxy center show that, for large $r$,

$$v_{\text{rot}}(r) \approx D = \text{const.}$$

With the Equation (c) classical approach, the observed flattening of rotational velocities implies for large $r$ values that

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As extensively described (often using Equation (c)), this $M(r)$ value is much larger than justified from the luminosity of stars and other methods. There are more sophisticated ways to estimate $M(r)$ (c.f. [Binney and Tremaine, 2008, Sec. 2.2.6–7], but Equation (c) is a standard approach.

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- Independent of the $m_j$ values, a spider-web solution behaves like a rotating rigid body, therefore the rotational velocity of a body with distance $r$ to the center of mass is

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- Now, apply Eq (c): $M(r) = \frac{r v_{\text{rot}}^2}{G} = \frac{D}{G} r^3.$
- This means that the predicted mass distribution is always given by a cubic equation *even though mass values have yet to be selected*!

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- In other words, rather than being cubic, the precise $M(r)$ distribution is, at best, linear.
- Next, we shall see some actual (numerically obtained) spacing between the rings of equal masses [Hénot and Rousseau, 2019] and later their mass distribution.

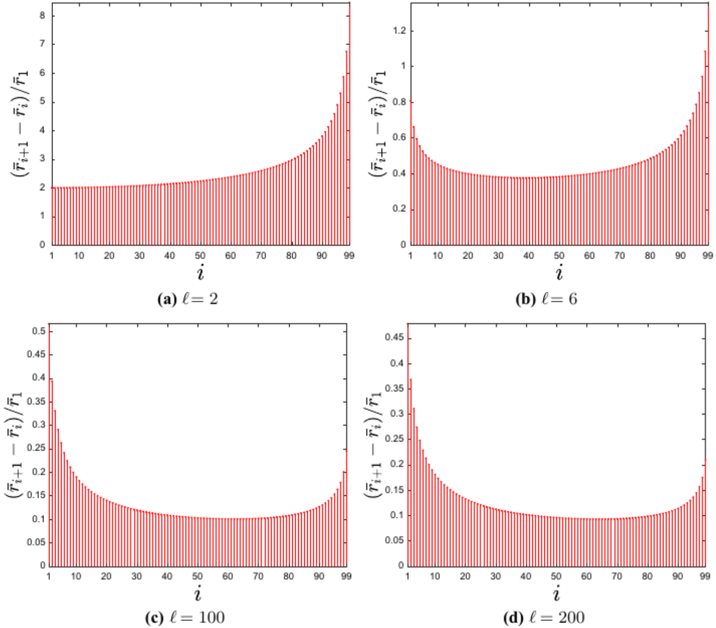


**Fig. 3** Spacing between consecutive circles of spiderweb central configurations with circles of equal mass and $\lambda = -1$, $n = 100$, $m_0 = 0$ for different values of $\ell$

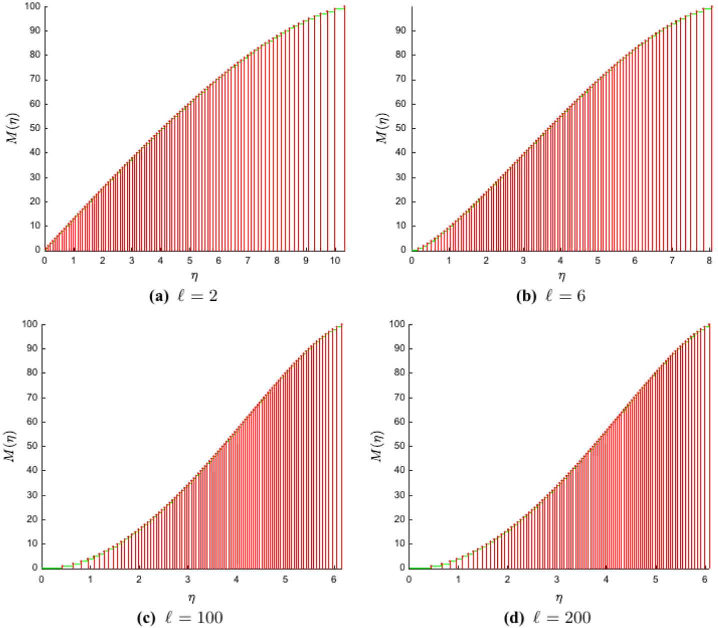


**Fig. 5** Mass distribution $M(\eta)$ of spiderweb central configurations with $\lambda = -1$, $n = 100$, $(m_0, m) = (0, 1/\ell, \ldots, 1/\ell)$ and different values of $\ell$

## What is wrong? [Saari, 2015b]

1. Newton's laws are incorrect. At least, they fail over the thousands of light-year distances in a galaxy; an attracting force stronger than Newton's law is needed.
2. The observed mass is correct, but it is insufficient to sustain the observed rotational velocities. With these larger velocities, expect the galaxy to fly apart.
3. The difference between the prediction and the known amount of mass is there; it just cannot be seen. The mass difference is due to unobserved dark matter.
4. The derivation of equation (c) is incorrect for systems of $N$ discrete bodies. With discrete systems, the equation (c) predictions can be seriously exaggerated.

# References

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