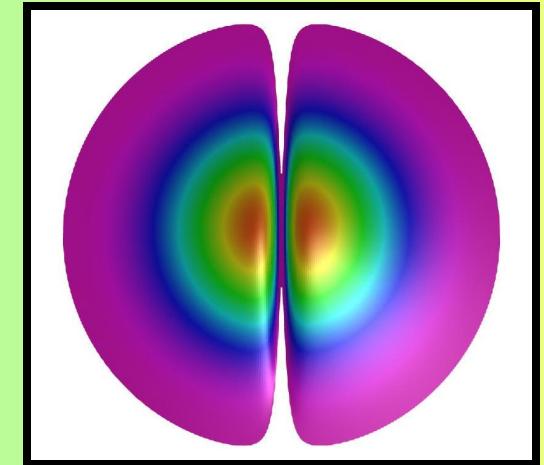
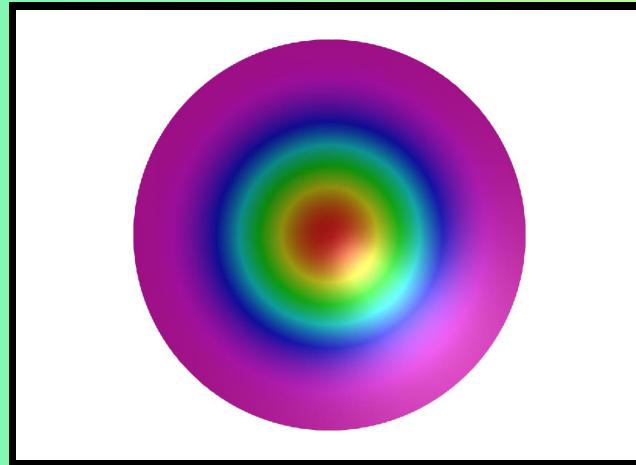
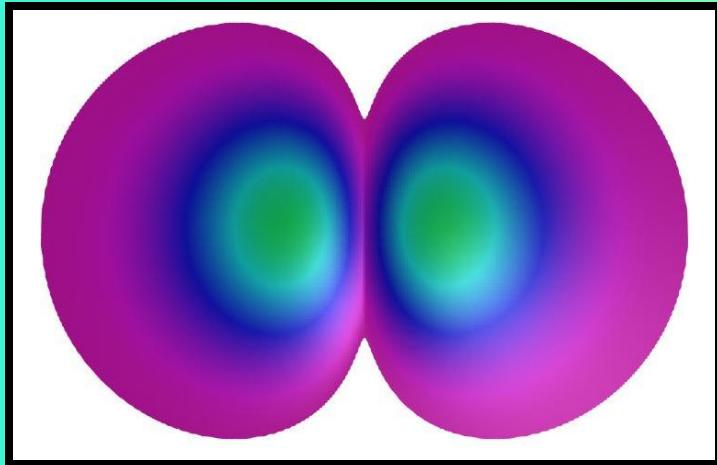


ANALYTICAL APPROXIMATION
FOR
DIFFERENTIALLY ROTATING BAROTROPES



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FIRST SELF-GRAVITATING ROTATING BODIES

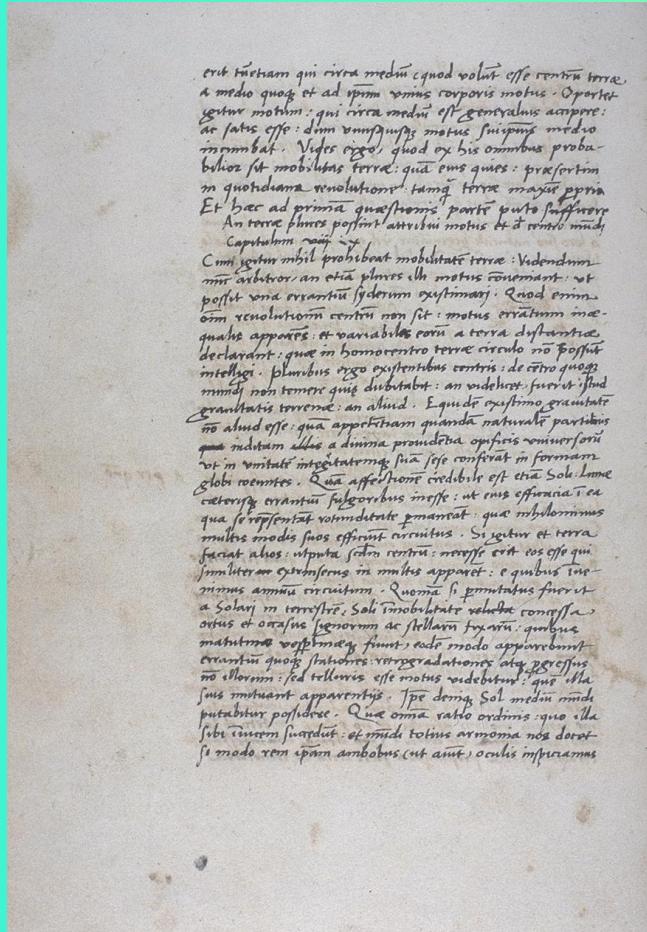


- C. Maclaurin, *Treatise on fluxions*, 1742
- Maclaurin spheroid theory
- Newton's "infinitesimal calculus" first application
- Reply to Bishop G. Berkely *The analyst: A DISCOURSE Addressed to an Infidel Mathematician*

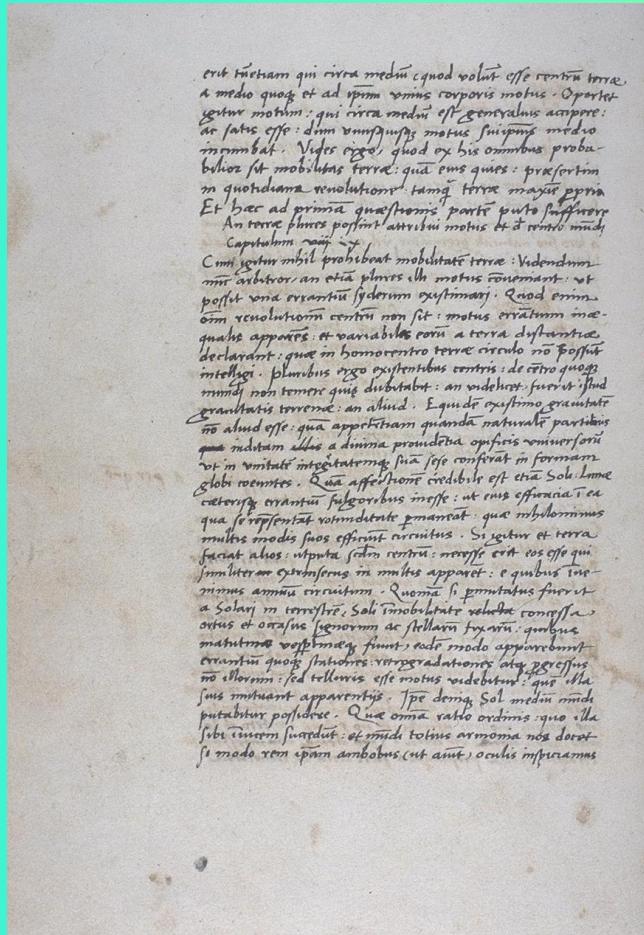
But what about non-rotating self gravitating bodies?

It is believed that Newton explained spherical shape of celestial bodies ...

Nicolaus Copernicus „*De Revolutionibus*”, Book I, Chapter IX:



Nicolaus Copernicus „*De Revolutionibus*”, Book I, Chapter IX:



For my part I believe that gravity is nothing but a certain natural desire, which the divine providence of the Creator of all things has implanted in parts, to gather as a unity and a whole by combining in the form of a globe. This impulse is present, we may suppose, also in the sun, the moon, and the other brilliant planets, so that through its operation they remain in that spherical shape which they display.

Copernicus also was motivated by relation of the gravity and shape of the celestial bodies

ROTATING BAROTROPES

Simple self-gravitating bodies:

- Barotropic EOS $p = p(\rho)$
- Newtonian self-gravity $\Phi_g = \Phi_g(\rho)$
- Time-independent (stationary) solutions $\rho = \rho(\mathbf{r})$
- No other important properties (magnetic fields, viscosity, etc.)

BAROTROPES: EOS EXAMPLES

1. Polytropic stars:

$$p = K\rho^\gamma = K\rho^{1+1/n}$$

2. Cold white dwarfs: *degenerate electron gas* EOS

3. Isothermal interstellar gas clouds:

$$p = c_s^2 \rho$$

4. Uniform density bodies: $\gamma \rightarrow \infty$

PURE ROTATION ASSUMPTION

We assume motion in our star in a form of *simple rotation*:

$$\mathbf{v} = r \Omega(r, z) \mathbf{e}_\phi$$

in cylindrical coords:

$$\mathbf{r} = (r, \phi, z)$$

and substitute into Euler and continuity equations:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla p - \nabla \Phi_g$$

$$\frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{v}) = 0$$

SELF-GRAVITATING, ROTATING GAS IN
FULL MECHANICAL EQUILIBRIUM

$$r \Omega(r, z)^2 \mathbf{e}_r = \frac{1}{\rho} \nabla p + \nabla \Phi_g$$

$$\frac{\partial \rho}{\partial t} + \Omega(r, z) \frac{\partial \rho}{\partial r} = 0$$

Continuity Equation has a general solution:

$$\rho(r, z, \phi; t) = F(r, z, \phi - \Omega t), \quad F - \text{arbitrary function}$$

$$\frac{\partial \rho}{\partial t} = 0 \leftrightarrow \text{axial symmetry}$$

Equatorial symmetry also can be proved (Lichtenstein theorem)

INTEGRABILITY CONDITION

$$\nabla \times (r \Omega(r, z)^2 \mathbf{e}_r) = \nabla \times \left(\frac{1}{\rho} \nabla p + \nabla \Phi_g \right)$$

$$2 r \Omega \frac{\partial \Omega}{\partial z} \mathbf{e}_\phi = \nabla \left(\frac{1}{\rho} \right) \times \nabla p$$

But $p = p(\rho)$:

$$\nabla \left(\frac{1}{\rho} \right) \times \nabla p = -\frac{1}{\rho^2} \frac{\partial p}{\partial \rho} \nabla p \times \nabla p \equiv 0 \text{ so:}$$

$$\frac{\partial \Omega(r, z)}{\partial z} = 0 \leftrightarrow \Omega = \Omega(r)$$

SUMMARY OF THE ROTATING BAROTROPS

Under the following assumptions:

- Our body is self-gravitating
- EOS is barotropic
- Pure rotation is the only movement allowed

we have found the following properties of the solutions of Euler equations:

- Angular velocity is constant over cylinders
- Density distribution is axially and equatorially symmetric and time-independent
- Density satisfies “**Rotating barotrope**” equation

“ROTATING BAROTROPE” EQUATION

h - enthalpy $\nabla h = \nabla p/\rho$,
 Φ_c - centrifugal potential:

$$h(\rho) + \Phi_g + \Phi_c = C$$

$$\Phi_c = \int_0^r r \Omega(r)^2 dr$$

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$$\Delta\Phi_g = 4\pi G\rho$$

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$$\Delta \Phi_g = 4\pi G \rho$$

$$\Phi_g = -G \int \frac{\rho dV}{|\mathbf{r} - \mathbf{r}'|}$$

CANONICAL FORM OF INTEGRAL
EQUATION

Hammerstein, A. 1930 Acta Mathematica, 54, 117-176

$$h(\rho) + \mathcal{R}(\rho) + \Phi_c = C$$

$$f = \mathcal{R}[F(f)]$$

where:

$$f = C - \Phi_c - h(\rho), F(f) = h^{-1}(f + \Phi_c - C)$$

SOLUTION METHOD

$$f_1 = \mathcal{R}[F(f_0)],$$

$$f_2 = \mathcal{R}[F(f_1)],$$

...

$$f_n = \mathcal{R}[F(f_{n-1})]$$

...

Iteration successfully applied numerically:

Self-consistent field method (Ostriker, J.P., Mark, J.W.-K. 1968 ApJ, **151**, 1075)

HSCF (Hachisu, I. 1986 ApJS, **61**, 479)

FIRST-ORDER APPROXIMATION

$$f_1 = \mathcal{R}[F(f_0)],$$

$$f_2 = \mathcal{R}[F(f_1)],$$

...

$$f_n = \mathcal{R}[F(f_{n-1})]$$

...

FIRST-ORDER APPROXIMATION

$$\begin{aligned} f_1 &= \mathcal{R}[F(f_0)], & \longrightarrow & h(\rho_1) = -\mathcal{R}(\rho_0) - \Phi_c + C \\ f_2 &= \mathcal{R}[F(f_1)], \\ &\dots \\ f_n &= \mathcal{R}[F(f_{n-1})] \\ &\dots \end{aligned}$$

FIRST-ORDER APPROXIMATION

$$\begin{aligned} f_1 &= \mathcal{R}[F(f_0)], \\ f_2 &= \mathcal{R}[F(f_1)], \\ &\dots \end{aligned} \qquad \longrightarrow \qquad h(\rho_1) = -\mathcal{R}(\rho_0) - \Phi_c + C$$

Let ρ_0 is density for non-rotating star $\Phi_c \equiv 0$:

$$\begin{aligned} f_n &= \mathcal{R}[F(f_{n-1})] \\ &\dots \end{aligned} \qquad h(\rho_0) + \mathcal{R}(\rho_0) = C_0$$

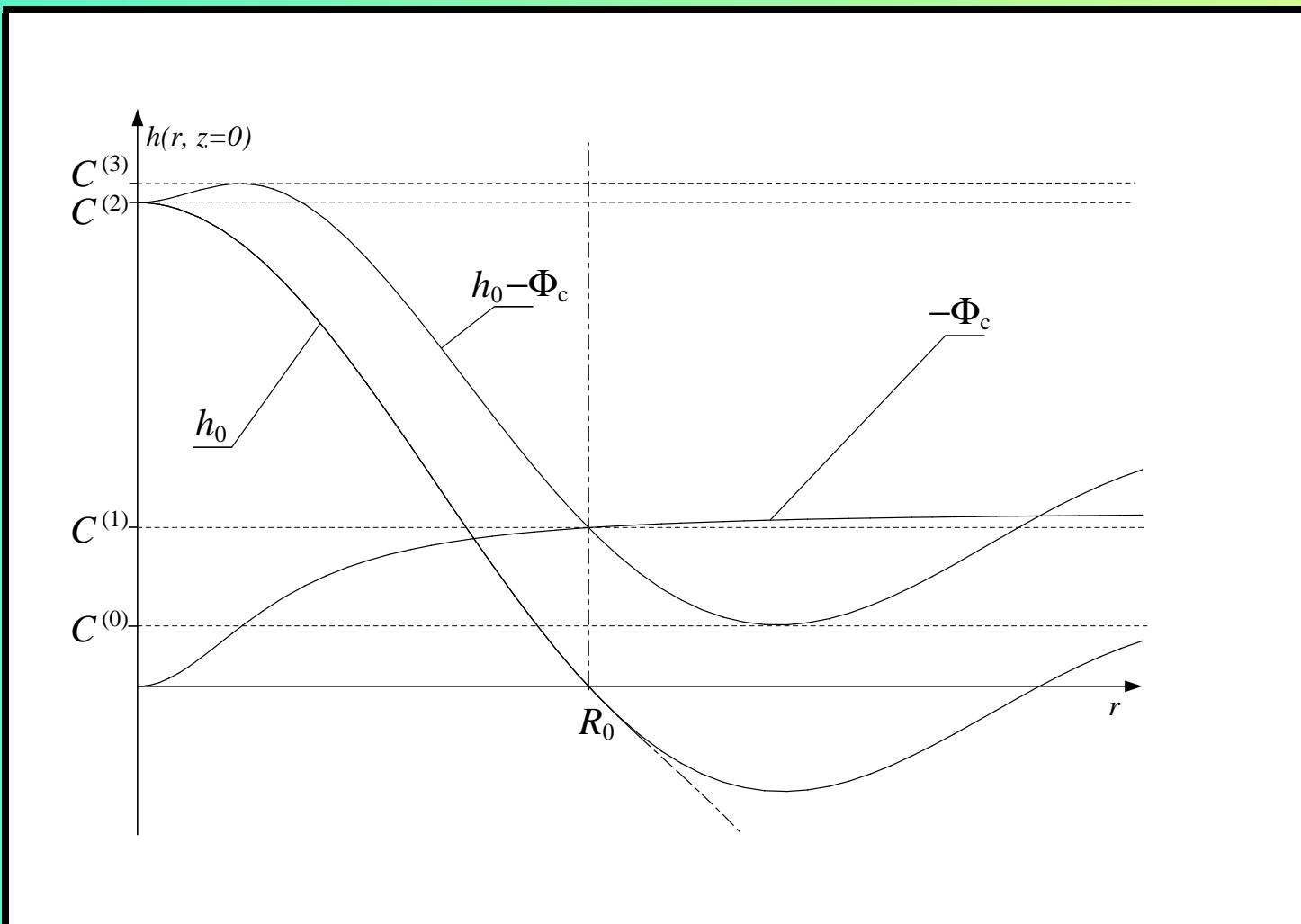
FIRST-ORDER APPROXIMATION

$$\begin{aligned}
 f_1 &= \mathcal{R}[F(f_0)], & h(\rho_1) &= -\mathcal{R}(\rho_0) - \Phi_c + C \\
 f_2 &= \mathcal{R}[F(f_1)], & & \\
 &\dots & \text{Let } \rho_0 \text{ is density for non-rotating star } \Phi_c \equiv 0: \\
 f_n &= \mathcal{R}[F(f_{n-1})] & h(\rho_0) + \mathcal{R}(\rho_0) &= C_0 \\
 &\dots
 \end{aligned}$$

No
integration
at all!

$$h(\rho_1) = h(\rho_0) - \Phi_c + C - C_0$$

$$h_1 = h_0 - \Phi_c - \Delta C$$



VALUE OF ΔC

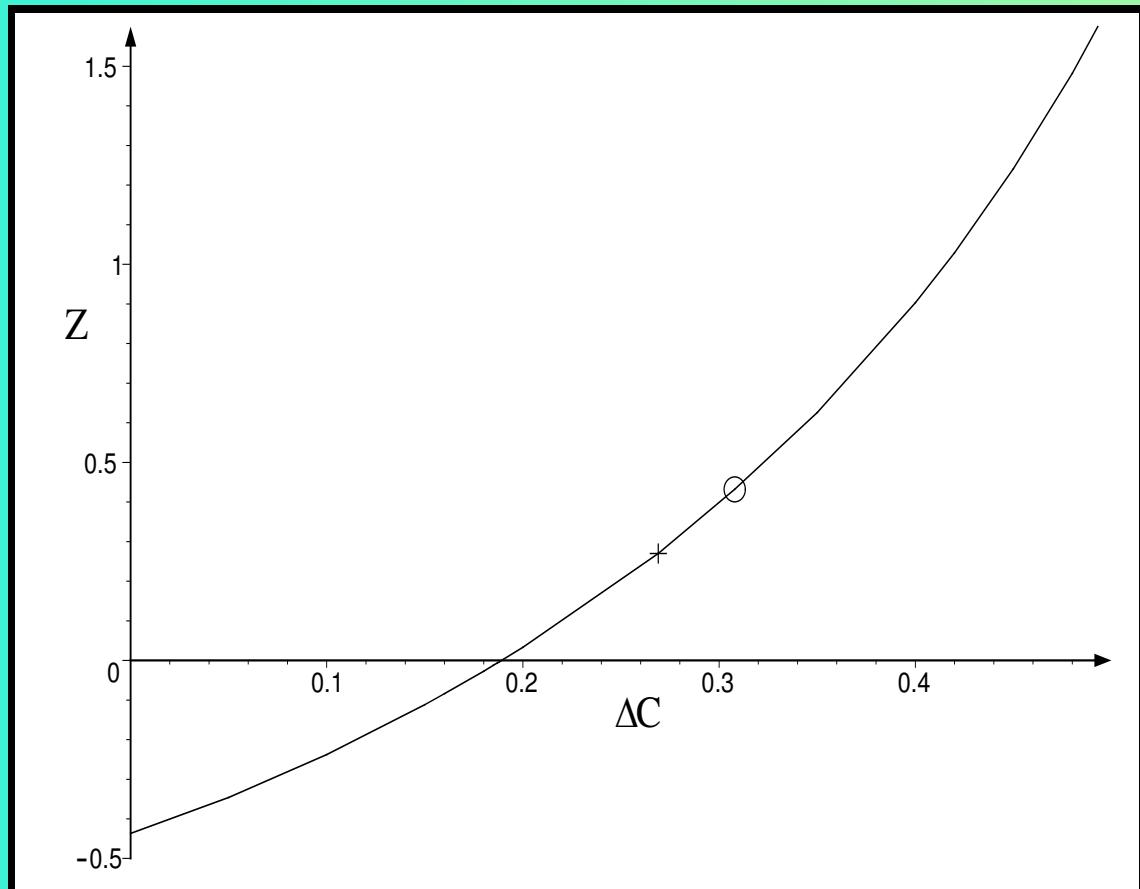
By substitution of our formula into basic equation we get:

$$\Delta C = \Phi_c(r)$$

This holds only if $\Delta C = 0, \Phi_c \equiv 0$. Instead, we can use mean value:

$$\Delta C = -\hat{\Phi}_c = - \left(\frac{4}{3} \pi R_0^3 \right)^{-1} \int_{V_0} \Phi_c d^3 \mathbf{r}$$

$$-\hat{\Phi}_c < C^{(1)} \equiv -\Phi_c(R_0)$$

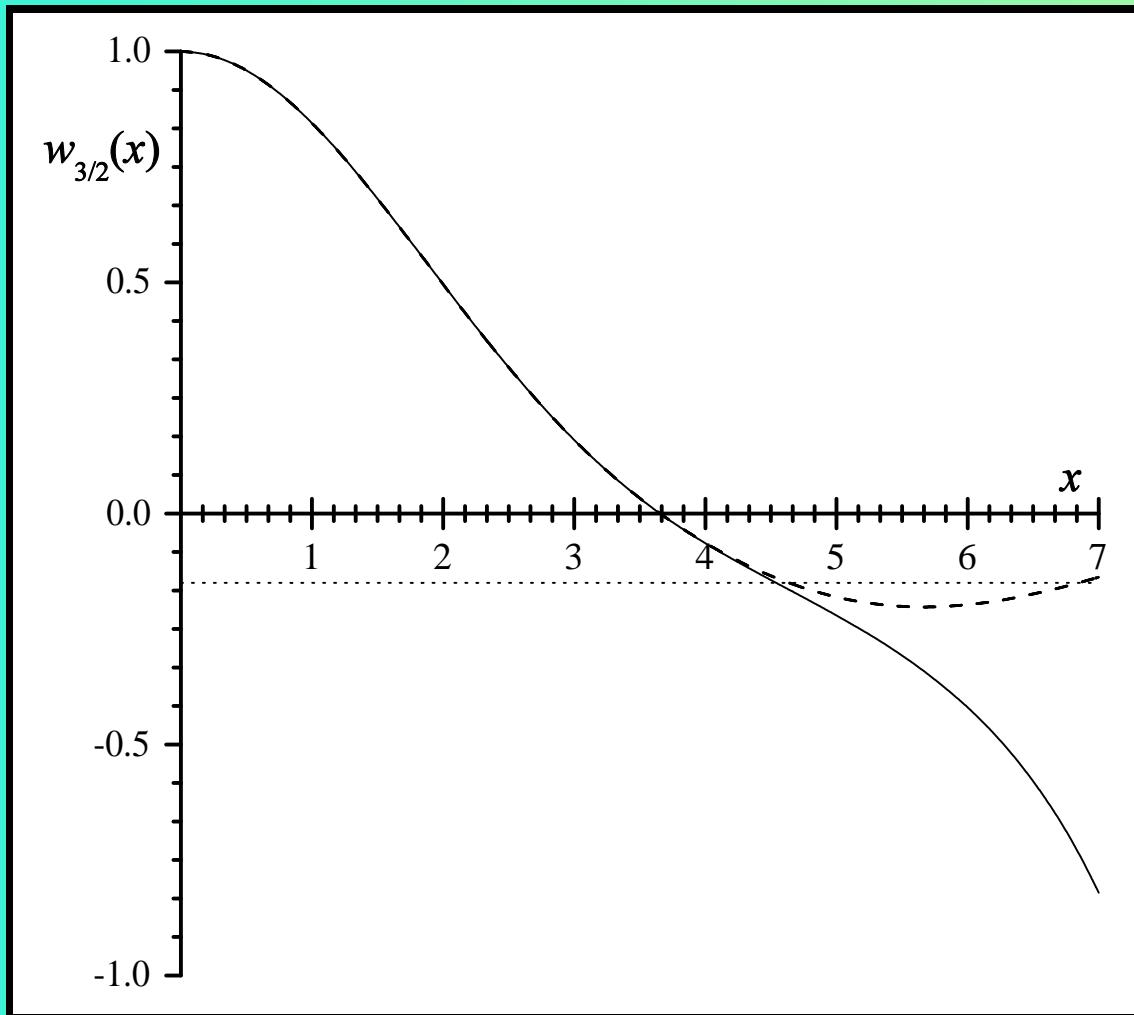
ΔC FROM VIRIAL THEOREM

$$\Delta C = -\Phi_c(R_0) \quad (\circ)$$

$$\Delta C = -\widehat{\Phi}_c \quad (\times)$$

$$Z = \frac{2 E_k - |E_g| + 3 \int p \, dV}{|E_g|}$$

NEGATIVE DENSITY (ENTHALPY)



To compute structure of rotating barotrope we need to know *unphysical* part of non-rotating solution with $\rho < 0$!

$$\dots + w^n$$

in Lane-Emden equation replaced by "odd-like" or "even-like" term:

$$\dots + |w|^n \quad \text{or} \quad \dots + \text{sign}(w)|w|^n$$

EXAMPLE: POLYTROPIC EOS

Enthalpy is:

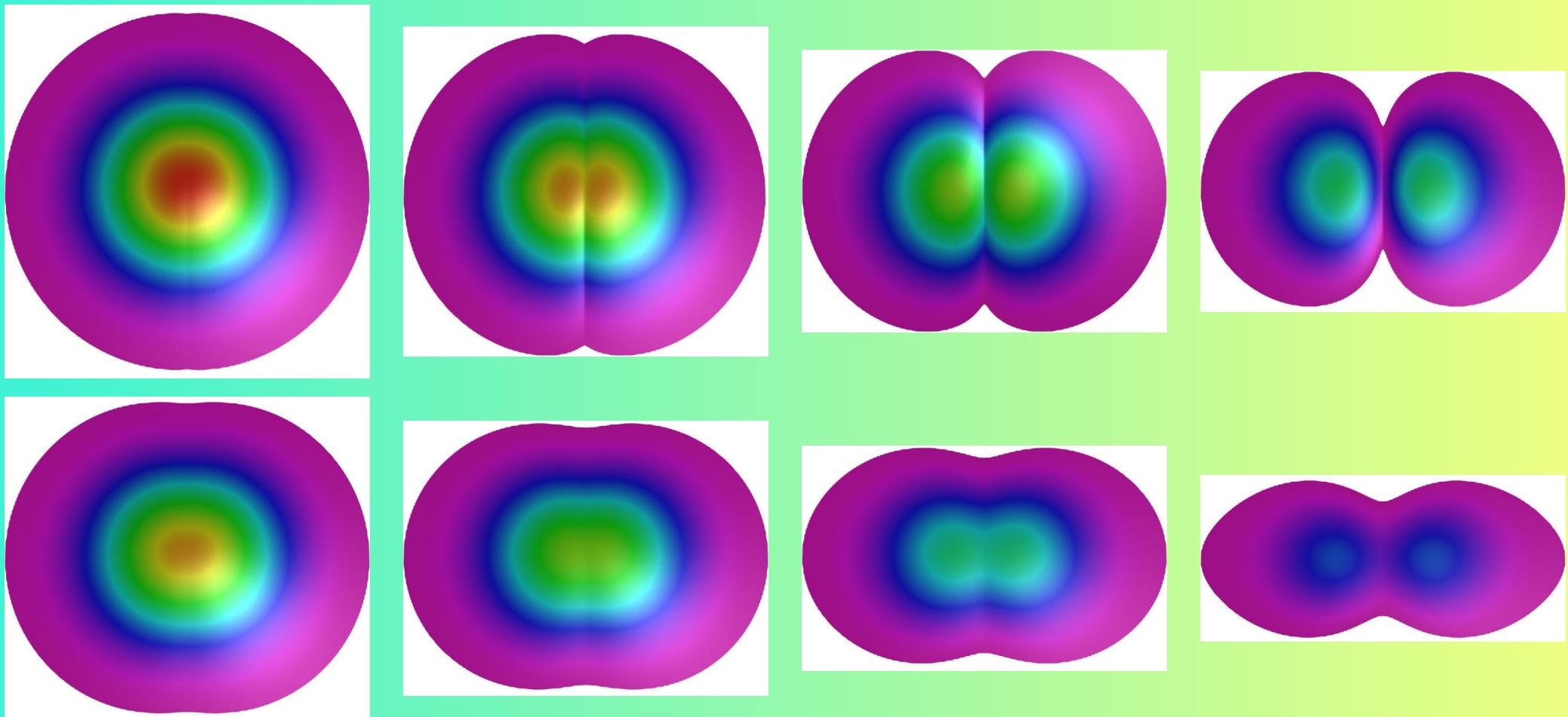
$$h(\rho) = \frac{K\gamma}{\gamma - 1} \rho^{\gamma-1}$$

Zero-order – n-th Lane-Emden function w_n :

$$\rho_0 = \rho_c (w_n)^n$$

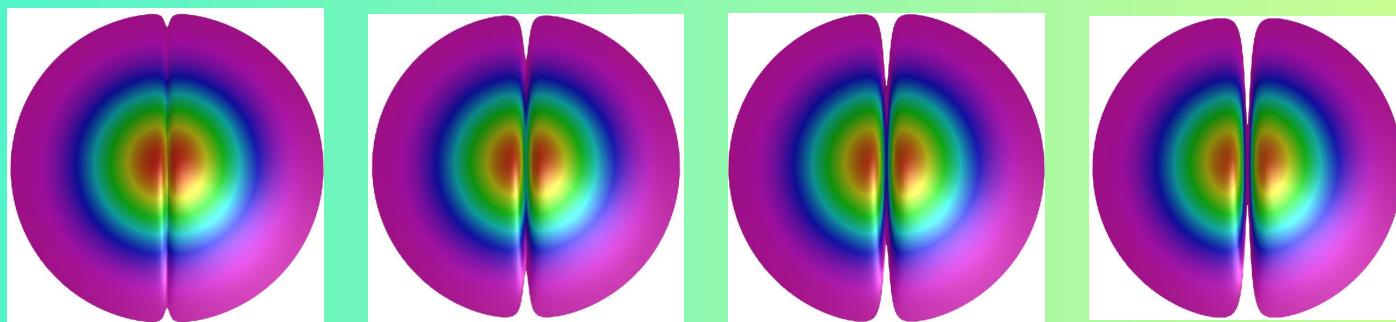
Approximate formula:

$$\rho_1 = \left[\rho_c^{1/n} w_n - \frac{1}{n K \gamma} (\Phi_c + \Delta C) \right]^n$$



Properties of the enthalpy distributions for $n = 3/2$ polytropic sequence with $v\text{-const}$ rotation law. Parameters of rotation are: upper row, from left: differentiability $A = 0.02R_0$; central angular velocity: $\Omega_0 = 1, 3, 5, 7$; lower row: differentiability $A = 0.2R_0$; central angular velocity: $\Omega_0 = 0.5, 0.75, 1.0, 1.25$.

A=0.02



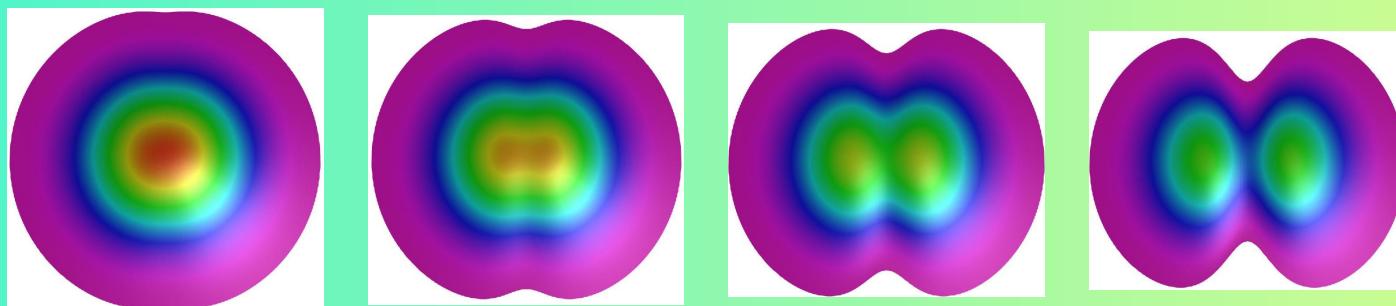
$\Omega_0 = 75$

$\Omega_0 = 150$

$\Omega_0 = 200$

$\Omega_0 = 250$

A=0.2



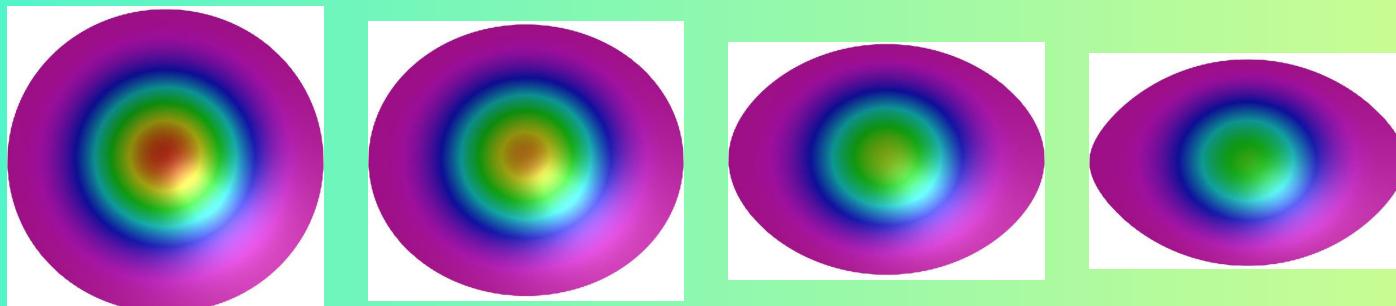
$\Omega_0 = 0.5$

$\Omega_0 = 1.0$

$\Omega_0 = 1.5$

$\Omega_0 = 2.0$

A=2



$\Omega_0 = 0.01$

$\Omega_0 = 0.02$

$\Omega_0 = 0.03$

$\Omega_0 = 0.035$

EXAMPLE: ELEMENTARY FUNCTIONS

For $n = 1$, $\Omega(r) = \Omega_0/(1 + r^2/A^2)$ and $\Delta C = \Phi_c(R_0 = \pi)$ we get:

$$\rho_1(r, z) = \frac{\sin \sqrt{r^2 + z^2}}{\sqrt{r^2 + z^2}} + \frac{1}{2} \frac{\Omega_0^2 A^2 r^2}{1 + \frac{r^2}{A^2}} - \frac{1}{2} \frac{\Omega_0^2 A^2 \pi^2}{1 + \frac{\pi^2}{A^2}}$$

Not very accurate, but we have purely analytical formula for differentially rotating barotrope!

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Not very accurate, but we have purely analytical formula for differentially rotating barotrope!

As far as I am aware, no single problem, not even a stability problem, has been solved in a differentially rotating self-gravitating medium. Even without magnetic fields, and even linearizing the equations, it is very hard to make progress.

Prendergast 1962, p.318

OFF-CENTER DENSITY (ENTHALPY) MAXIMUM

Near $r = 0$ we have:

$$w_n(x) \simeq 1 - \frac{1}{6} x^2 + \frac{n}{120} x^4 + \dots, \quad \Phi_c \simeq -\frac{1}{2} \Omega_0^2 r^2 + \dots$$

enthalpy for $z = 0$ is approximately:

$$h_1(r) = h_c + \left(\Omega_0^2 - \frac{4}{3} \pi G \rho_c \right) r^2 + \dots$$

If $\Omega_0^2 > \frac{4}{3} \pi G \rho_c$ to $\rho_c < \rho_{max}$!

In Carbon-Oxygen white dwarfs, ignition density $\rho_c \simeq 2 \cdot 10^9$ gives $\Omega_0 \sim 25$ rad/s.

$$\Omega_0(\text{Fe}) > 30 \text{ rad/s}$$

Heger & Langer 2000:

$$\Omega_0(\text{He}) \sim 10^{-3} \text{ rad/s}$$

Off-center ignition of Ia supernova unlikely

FIRST-ORDER APPROXIMATION
versus NUMERICAL RESULTS

Example: polytrope with $n = 3/2$ and j -const rotation law defined by $A = 0.2R_0$ and $\Omega_0 = 1.5$.

Solid line — numerical results of Eriguchi & Muller (1985).

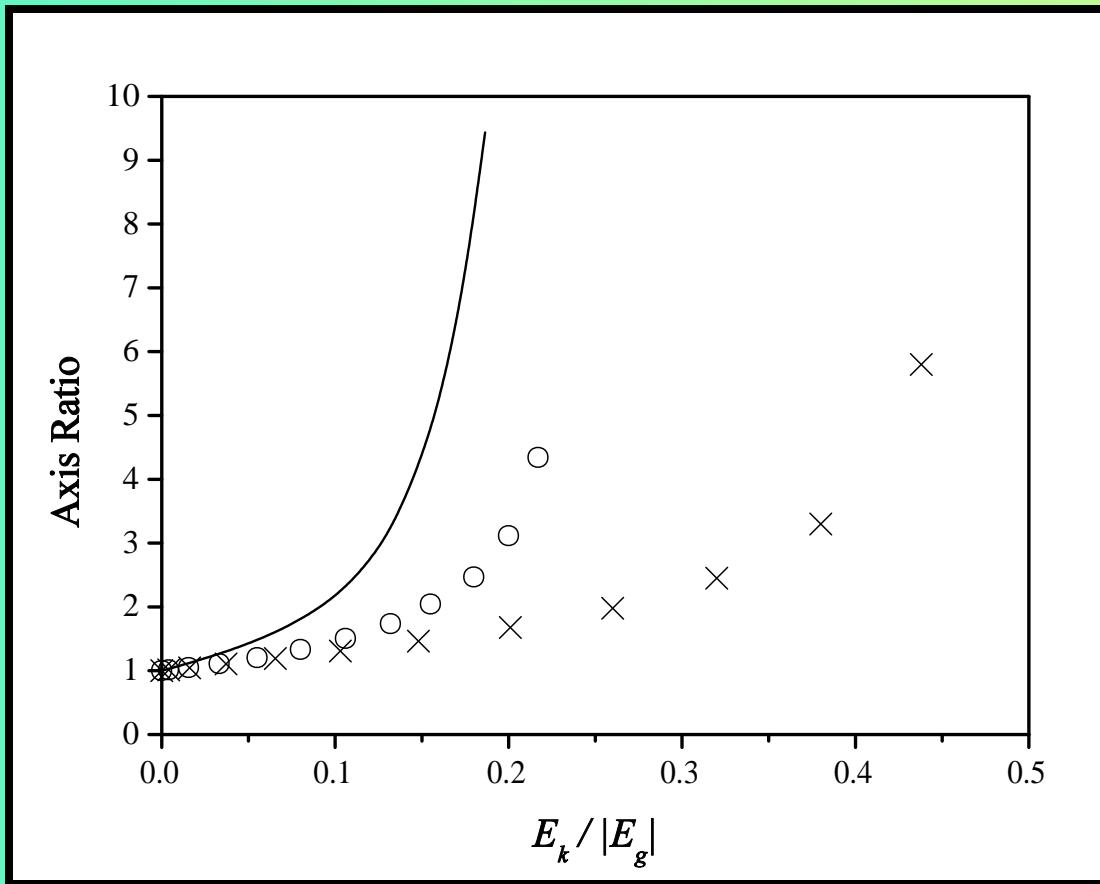
Integral equation solved on finite grid.

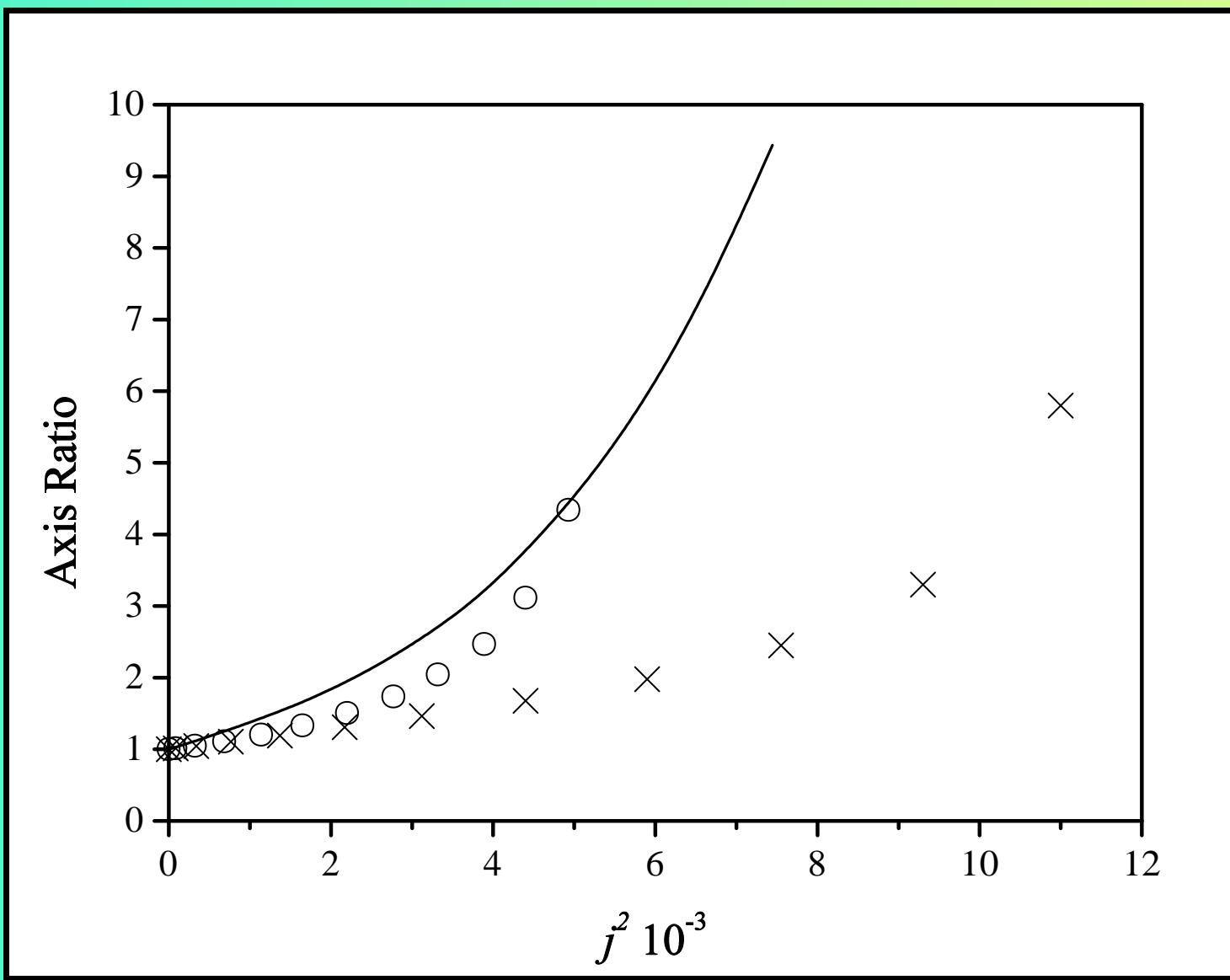
× — analytical formula with

$$\Delta C = -\hat{\Phi}_c = - \left(\frac{4}{3} \pi R_0^3 \right)^{-1} \int_{V_0} \Phi_c d^3 \mathbf{r}$$

○ — ΔC choosen to satisfy virial theorem

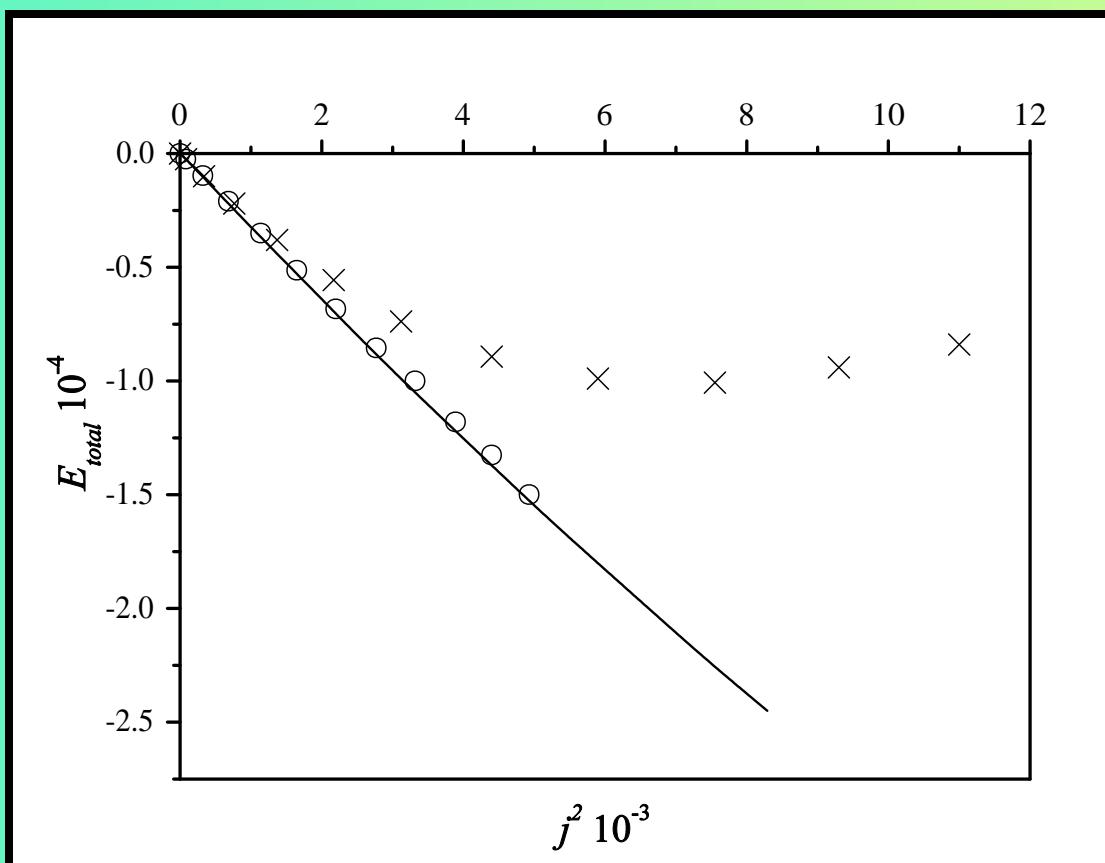
AXIS RATIO TESTS





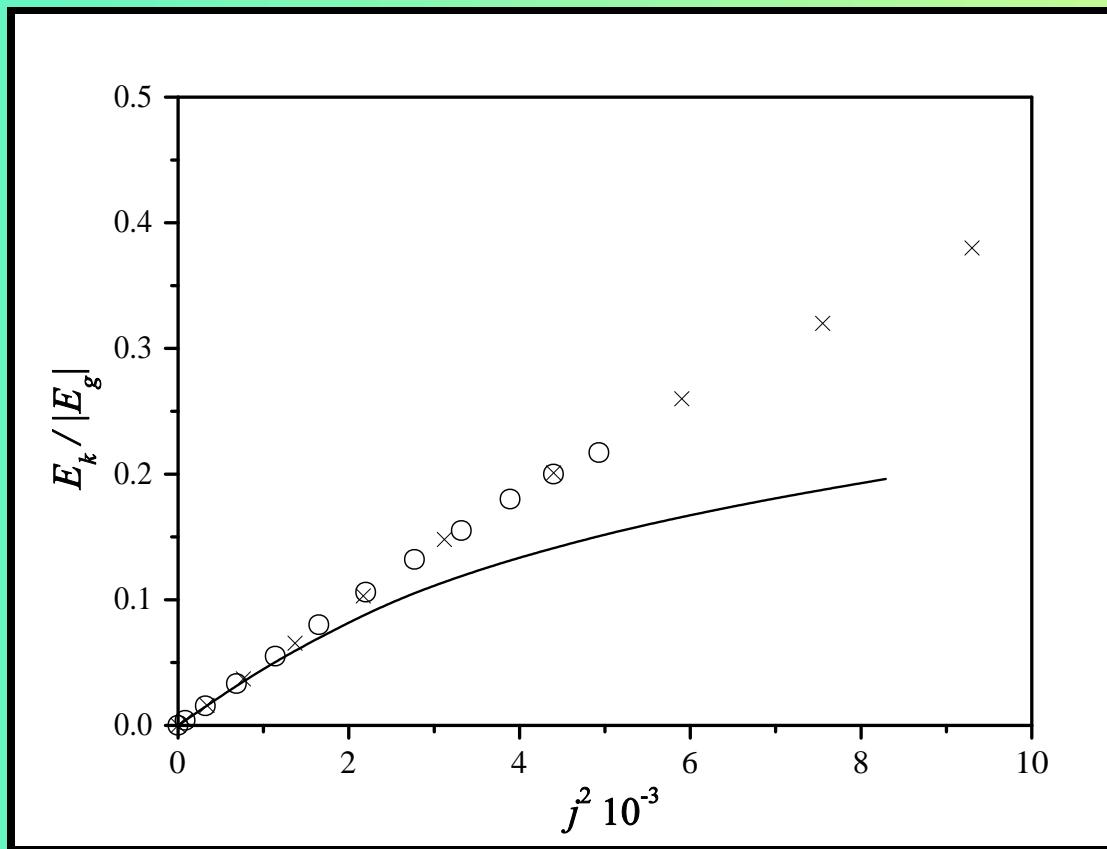
TOTAL ENERGY TESTS

$$E_{tot} = (E_k + E_g + U)/E_0$$



DIMENSIONLESS ANGULAR MOMENTUM

$$j^2 = \frac{1}{4\pi G} \frac{J^2}{M^{10/3}} \rho_{max}^{1/3}$$



POSSIBLE APPLICATIONS & EXTENSIONS

1. Second order approximation \equiv *New numerical scheme*
2. Initial guess for numerical algorithms
3. Fitting formula for numerically obtained enthalpy
4. Semi-analytical sequences with constant mass and/or angular momentum
5. Educational/ lecture tool

OPEN QUESTIONS:

1. Analytical formula for ΔC ($\sim \Omega_0^{2.18}$ from V.T. numerically)
2. Unique analytical continuation beyond $\rho = 0$ for non-rotating component

END

THE j -CONST ANGULAR VELOCITY PROFILE

$$\Omega(r) = \frac{\Omega_0}{1 + (r/A)^2}.$$

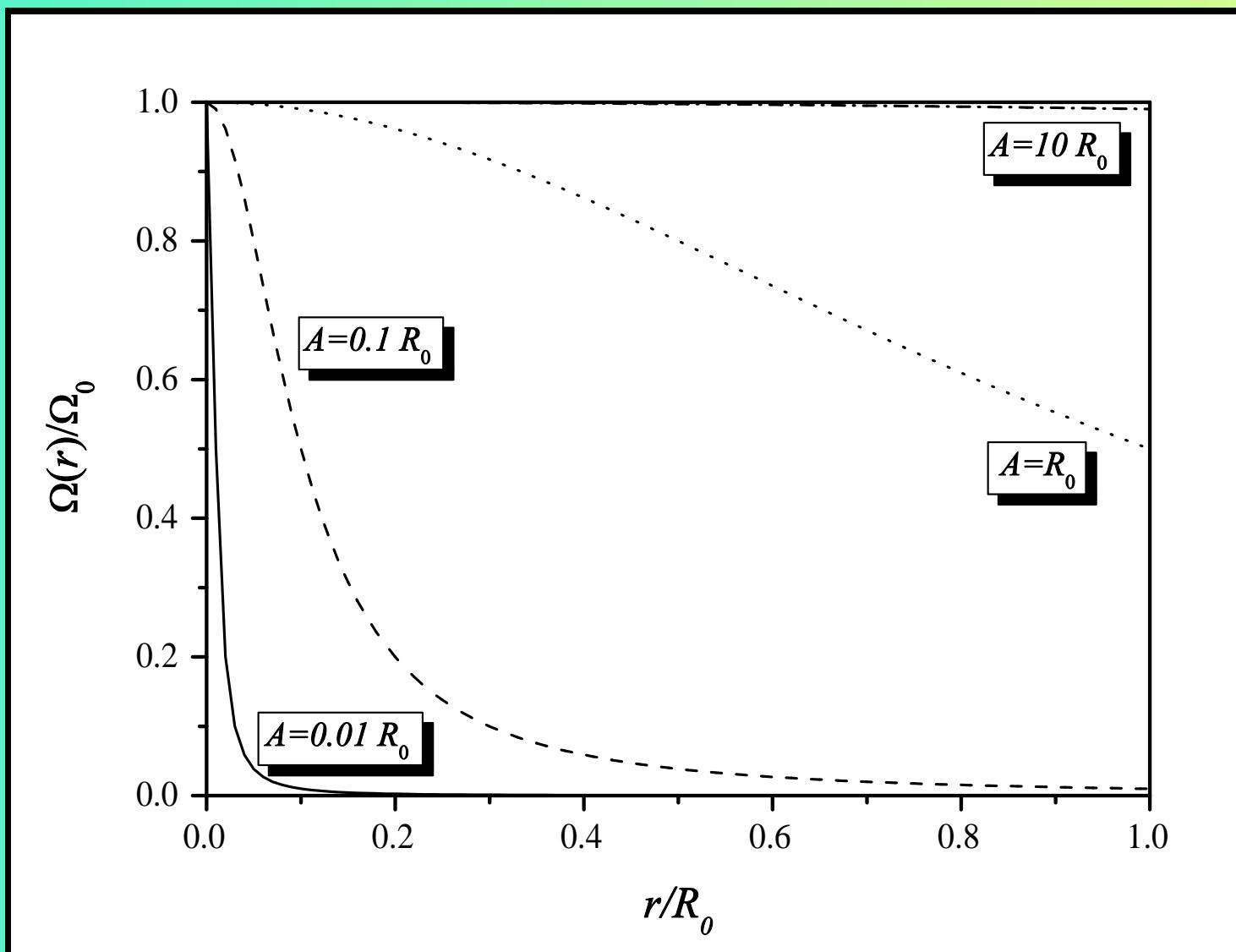
$$\Phi_c(r) = -\frac{1}{2} \frac{\Omega_0^2 r^2}{1 + (r/A)^2}.$$

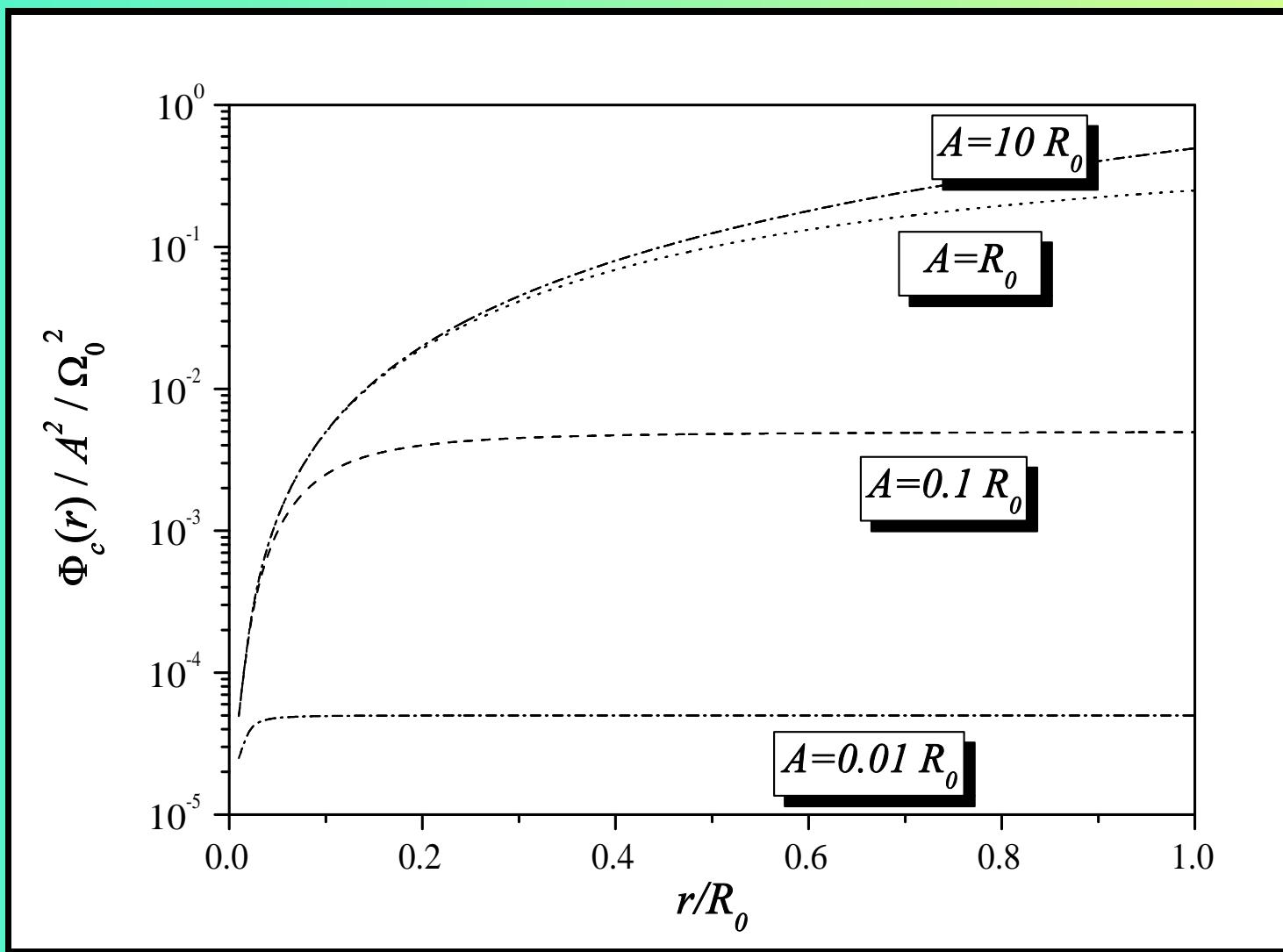
The name j -const reflects the behaviour for $A \rightarrow 0$:

$$\Omega(r) = \frac{A^2 \Omega_0}{A^2 + r^2} \sim \frac{A^2 \Omega_0}{r^2}.$$

Specific angular momentum is defined as $j = \rho \Omega(r) r^2$. Therefore $\Omega(r)$ behaves as for rotating body with $j = const$.

If $A \rightarrow \infty$ then $\Omega(r) \rightarrow \Omega_0$ it corresponds to the uniform rotation.





THE v -CONST ROTATION LAW

$$\Omega(r) = \frac{\Omega_0}{1 + r/A}.$$

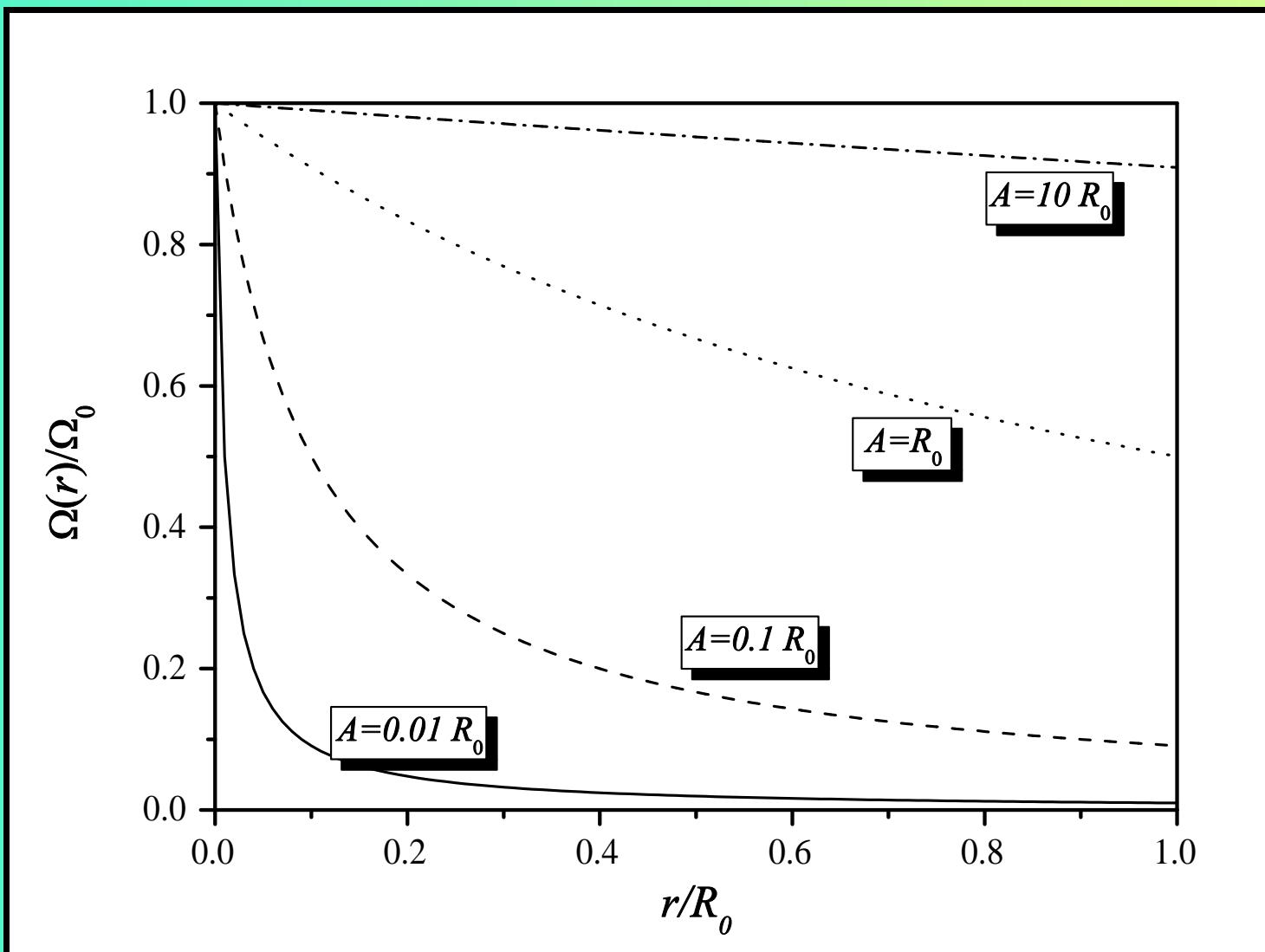
$$\Phi_c(r) = -\Omega_0^2 A^2 \left[\frac{1}{1 + A/r} - \ln(1 + r/A) \right].$$

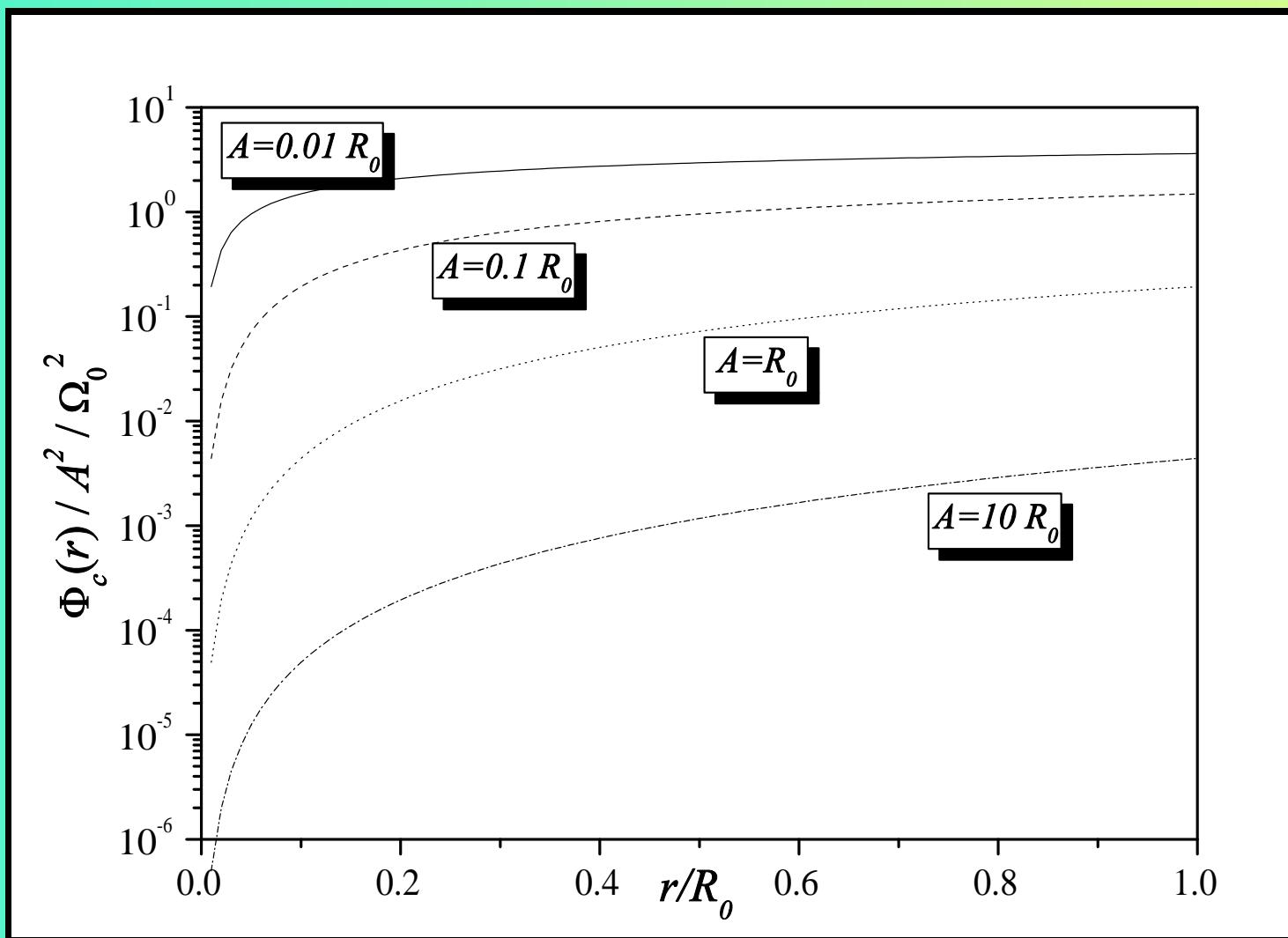
Similarly to the case described in the previous subsection, the name v -const reflects the behaviour for $A \rightarrow 0$:

$$\Omega(r) = \frac{A\Omega_0}{A + r} \sim \frac{\Omega_0}{r}.$$

Accordingly, because of the relation $v = \Omega(r) r$ between angular and linear velocity, $\Omega(r)$ behaves as for matter rotating with constant linear velocity v .

Again, if $A \rightarrow \infty$ then $\Omega(r) \rightarrow \Omega_0 = const.$





GRAVITATIONAL POTENTIAL INSIDE
ELLIPOSOID

$$\Phi_g(x, y, z) = \pi G \rho [(a_x^2 - x^2) A_x + (a_y^2 - y^2) A_y + (a_z^2 - z^2) A_z]$$

$$A_i = a_x a_y a_z \int_0^\infty \frac{du}{(a_i^2 + u)(a_x^2 + u)(a_y^2 + u)(a_z^2 + u)}$$

$$\Phi_c(x, y, z) = \frac{1}{2} \Omega^2 (x^2 + y^2)$$

$\Phi_g + \Phi_c = C = \text{const} \longrightarrow$ ellipsoid equation.

SYSTEM OF ALGEBRAIC EQUATIONS

$$(1) \frac{\tilde{C}}{a_x^2} = \pi G \rho A_x - \frac{1}{2}\Omega^2$$

$$(2) \frac{\tilde{C}}{a_y^2} = \pi G \rho A_y - \frac{1}{2}\Omega^2$$

$$(3) \frac{\tilde{C}}{a_z^2} = \pi G \rho A_z$$

$$(4) \tilde{C} = \pi G \rho (a_x^2 A_x + a_y^2 A_y + a_z^2 A_z) - C$$

$$(5) V = \frac{4}{3} \pi a_x a_y a_z$$

Solutions of this system $(a_x, a_y, a_z, C, \tilde{C})$ is a function of parameters (V, ρ, Ω) .

Let us denote: $a_x/a_z \equiv \varepsilon$ and $\chi = \frac{\Omega^2}{2\pi G \rho}$

MACLAURIN SPHEROID: THE SHAPE

$$\chi = \frac{\varepsilon(1 + 2\varepsilon^2) \arccos \varepsilon - 3\varepsilon^2 \sqrt{1 - \varepsilon^2}}{(1 - \varepsilon^2)^{3/2}}$$

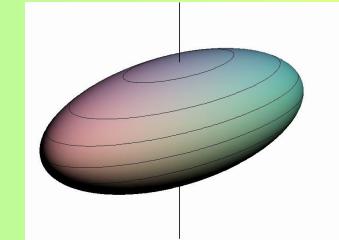
- Dla $\frac{E_k}{E_g} = 0 \longrightarrow$ ball
- Dla $0 < \frac{E_k}{E_g} < 0.5 \longrightarrow$ Maclaurin sferoid
- Dla $\frac{E_k}{E_g} = 0.5 \longrightarrow$ infinite disk at rest

Virial theorem:

$$\frac{E_k}{E_g} = \frac{1}{2} - \int p d^3\mathbf{r}/E_g \longrightarrow 0 < \frac{E_k}{E_g} < \frac{1}{2}$$

JACOBI ELLIPSOID

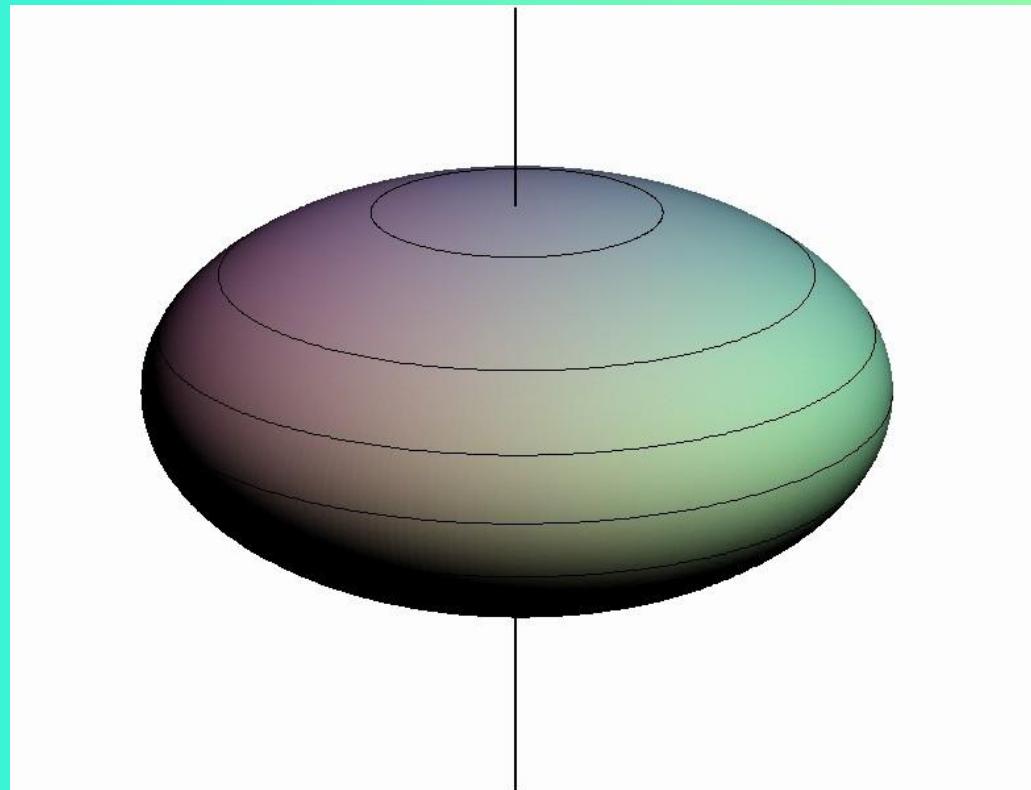
For $E_k/E_g > 0.1375$ ($\chi > 0.187$) non-axisymmetric solution exist! (C. Jacobi, 1834)



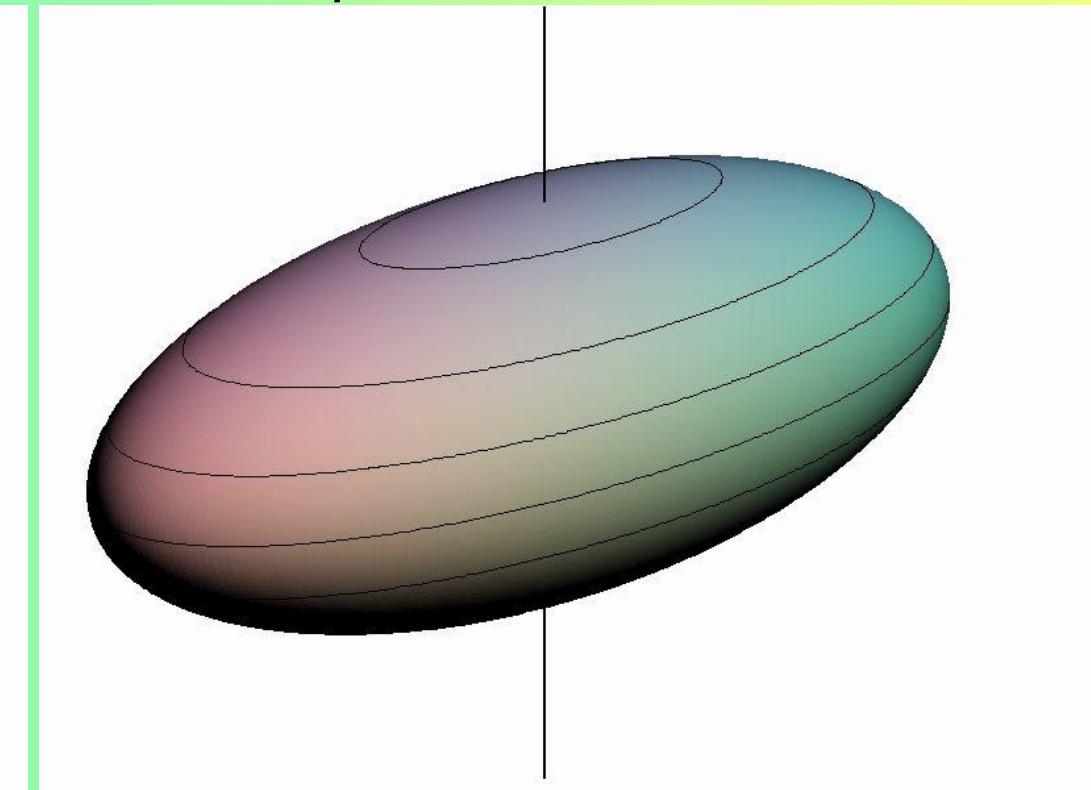
- For $\frac{E_k}{E_g} \leq 0.1375 \longrightarrow$ Maclaurin sferoid
- For $0.1375 < \frac{E_k}{E_g} < 0.5 \longrightarrow$ Jacobi ellipsoid (triaxial)
- For $\frac{E_k}{E_g} = 0.5 \longrightarrow$ infinite „rod” at rest

For exactly the same M & J Jacobi ellipsoid is „ground state”: $E_k + E_g = \min$

Maclaurin Sferoid



Jacobi Ellipsoid



$$\frac{E_k}{E_g} = 0.23$$

STABILITY

- $E_k/E_g < 0.1375 \longrightarrow$ Maclaurin spheroid is stable
- $E_k/E_g = 0.1375 \longrightarrow$ Maclaurin spheroid \equiv Jacobi Ellipsoid
- $0.1375 < E_k/E_g < 0.27 \longrightarrow$ Maclaurin spheroid is secularly unstable
- $E_k/E_g > 0.27 \longrightarrow$ Maclaurin s. is dynamically unstable
- $0.1375 < E_k/E_g < 0.2328 \longrightarrow$ Jacobi ellipsoid. is stable
- $E_k/E_g > 0.2328 \longrightarrow$ Jacobi e. is dynamically unstable

No stable configurations — no need for precise description of structure beyond 0.27 (or 0.2382).

ROCHE MODEL

Central mass M + envelope $\rho \rightarrow 0$.

Equipotential surface:

$$\frac{GM}{\sqrt{r^2 + z^2}} + \frac{1}{2}\Omega^2 r^2 = \text{const}$$

Critical surface:

$$\frac{GM}{R_e^2} = \Omega^2 R_e, \quad R_e = \left(\frac{GM}{\Omega^2} \right)^{1/3}$$

GENERALIZED JEANS MODEL

Maclaurin Sferoid with mass M and volume V_1 +
massless envelope $\rho \rightarrow 0$ and volume V_2 .

$$\bar{\rho} = \frac{M}{V_1 + V_2}, \quad \rho_M = \frac{M}{V_1}$$
$$\chi_R = \frac{\Omega^2}{2\pi G \bar{\rho}}, \quad \chi_M = \frac{\Omega^2}{2\pi G \rho_M}$$

$$\chi_R = 0.36, \quad \chi_M = 0.187$$

$$\rho_M/\bar{\rho} = \chi_R/\chi_M \simeq 2 \text{ co daje } n \simeq 0.6 \quad [0.83, 0.808]$$

GENERAL RELATIVITY & ROTATING BAROTROPES

$$ds^2 = (e^\nu)^2 dt^2 - (e^\mu)^2 (dr^2 + r^2 d\theta^2) - (e^\psi)^2 (\omega dt - d\phi)^2$$

gdzie ν, μ, ψ, ω to funkcje r i θ .

4-velocity: $u^\alpha = \frac{e^{-\nu}}{\sqrt{1-v^2}} [1, 0, 0, \Omega] = \frac{e^{-\nu}}{\sqrt{1-v^2}} (t^\alpha + \Omega \phi^\alpha),$

$v \equiv |\mathbf{v}| = e^{\psi-\nu} (\Omega - \omega).$

EOS: $p = p(\varepsilon)$; entalpia: $\nabla h = \nabla p / (\varepsilon + p)$

$$\nabla h + \nabla \nu + \nabla \ln \sqrt{1 - v^2} + \frac{v^2}{1 - v^2} \nabla \Omega / (\Omega - \omega) = 0$$