

ROTATING POLYTROPES

Simple self-gravitating bodies:

- barotropic EOS $p = p(\rho)$
- Newtonian gravity
- time-independent
- no other important properties

SIMPLE BUT NOT TRIVIAL:

1. Polytropic stars:

$$p = K \rho^\gamma$$

2. Cold white dwarfs – *degenerate electron gas* EOS

3. Isothermal interstellar gas clouds:

$$p = c_s^2 \rho$$

EULER AND CONTINUITY EQUATIONS

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla p - \nabla \Phi_g$$

$$\frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{v}) = 0$$

PURE ROTATION

We assume motion in our star in a form of *simple rotation*:

$$\mathbf{v} = r \Omega(r, z) \mathbf{e}_\phi$$

in cylindrical coords:

$$\mathbf{r} = (r, \phi, z)$$

SELF-GRAVITATING, ROTATING GAS IN FULL MECHANICAL EQUILIBRIUM

$$r \Omega(r, z)^2 \mathbf{e}_r = \frac{1}{\rho} \nabla p + \nabla \Phi_g$$

$$\frac{\partial \rho}{\partial t} + \Omega(r, z) \frac{\partial \rho}{\partial r} = 0$$

CONTINUITY EQUATION – SOLUTION

$$\rho(r, z, \phi; t) = F(r, z, \phi - \Omega t)$$

F – arbitrary function

$$\frac{\partial \rho}{\partial t} = 0 \leftrightarrow \text{axial symmetry}$$

INTEGRABILITY CONDITION

$$\nabla \times \left(r \Omega(r, z)^2 \mathbf{e}_r \right) = \nabla \times \left(\frac{1}{\rho} \nabla p + \nabla \Phi_g \right)$$

$$2r\Omega \frac{\partial \Omega}{\partial z} \mathbf{e}_\phi = \nabla \left(\frac{1}{\rho} \right) \times \nabla p$$

But $p = p(\rho)$: $\nabla \left(\frac{1}{\rho} \right) \times \nabla p = -\frac{1}{\rho^2} \frac{\partial p}{\partial \rho} \nabla p \times \nabla p \equiv 0$ so:

$$\frac{\partial \Omega(r, z)}{\partial z} = 0 \leftrightarrow \Omega = \Omega(r)$$

CENTRIFUGAL POTENTIAL

$$\Phi_c(r) = - \int_0^r \Omega(\tilde{r})^2 \tilde{r} d\tilde{r}$$

$$\nabla \Phi_c(r) = -r \Omega(r)^2 \mathbf{e}_r$$

ENTHALPY

$$h(\rho) = \int \frac{1}{\rho} dp$$

$$\nabla h(\rho) = \frac{1}{\rho} \nabla p$$

Integration constant is defined to be such that:

$$h(\rho = 0) = 0$$

Euler equation becomes sum of gradients:

$$\nabla [h(\rho) + \Phi_c + \Phi_g] = 0$$

with solution:

$$h(\rho) + \Phi_c + \Phi_g = C = const$$

“ROTATING STAR” EQUATION

$$h(\rho) + \Phi_c + \Phi_g = C = \text{const}$$

$$\Delta\Phi_g = 4\pi G \rho$$

$$\Phi_g(\mathbf{r}) = \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}'$$

CANONICAL FORM OF INTEGRAL EQUATION

Hammerstein, A. 1930 Acta Mathematica, **54**, 117-176

$$h(\rho) + \mathcal{R}(\rho) + \Phi_c = C$$

$$f = \mathcal{R}[F(f)]$$

where:

$$f = C - \Phi_c - h(\rho), F(f) = h^{-1}(f + \Phi_c - C)$$

SOLUTION METHOD

$$f_1 = \mathcal{R}[F(f_0)],$$

$$f_2 = \mathcal{R}[F(f_1)],$$

...

$$f_n = \mathcal{R}[F(f_{n-1})]$$

...

Iteration successfully applied numerically:

Self-consistent field method (Ostriker, J.P., Mark, J.W.-K. 1968 ApJ, **151**, 1075)

HSCF (Hachisu, I. 1986 ApJS, **61**, 479)

ZERO-ORDER APPROXIMATION

$$C - \Phi_c - h(\rho_1) = \mathcal{R}(\rho_0)$$

Using non-rotating ρ_0 :

$$h(\rho_0) + \mathcal{R}(\rho_0) = C_0$$

We can eliminate integral operator \mathcal{R} :

$$C - \Phi_c - h(\rho_1) = C_0 - h(\rho_0)$$

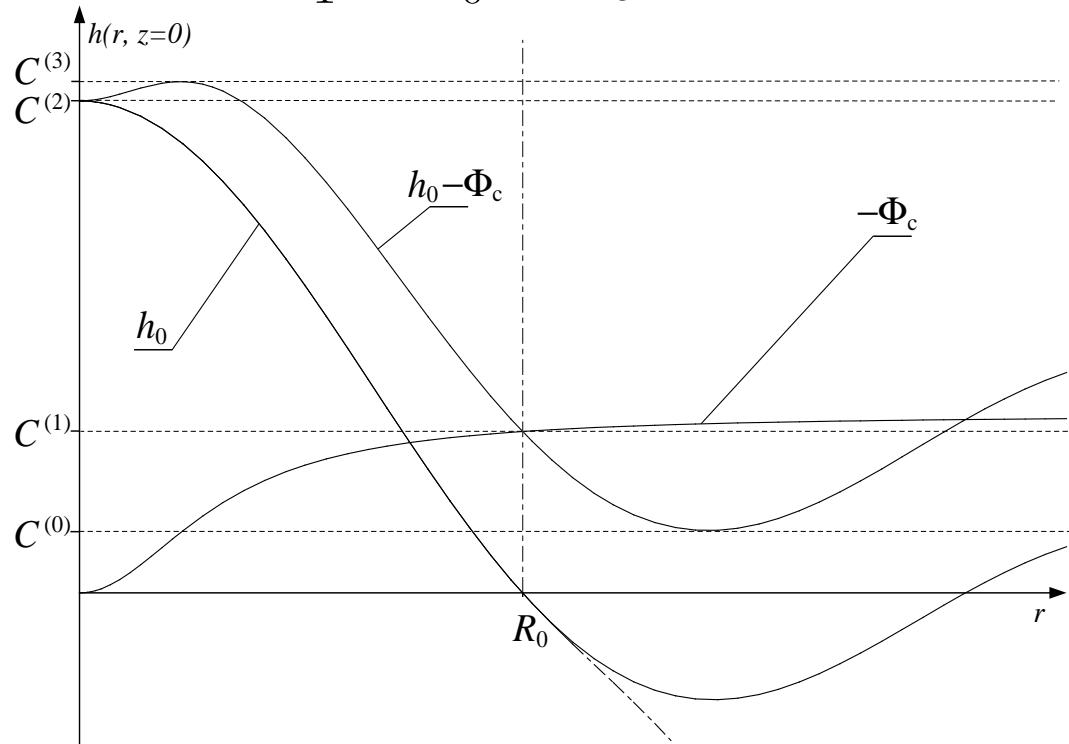
FIRST-ORDER APPROXIMATION

$$h(\rho_1) = h(\rho_0) - \Phi_c + C - C_0$$

or simpler, using enthalpy $h(\rho_0) \equiv h_0$, $h(\rho_1) \equiv h_1$:

$$h_1 = h_0 - \Phi_c + C - C_0$$

$$h_1 = h_0 - \Phi_c + \Delta C$$



VALUE OF ΔC

By substitution of our formula into basic equation we
get:

$$\Delta C = \Phi_c(r)$$

This holds only if $\Delta C = 0, \Phi_c \equiv 0$. Instead, we can use
mean value:

$$\Delta C = -\hat{\Phi}_c = - \left(\frac{4}{3} \pi R_0^3 \right)^{-1} \int_{V_0} \Phi_c d^3 \mathbf{r}$$

VALUE OF ΔC

We can avoid negative enthalpy with:

$$\Delta C = -\Phi_c(R_0)$$

EXAMPLE: POLYTROPIC EOS
Enthalpy is:

$$h(\rho) = \frac{K\gamma}{\gamma - 1} \rho^{\gamma-1}$$

Zero-order – n-th Lane-Emden function w_n :

$$\rho_0 = \rho_c (w_n)^n$$

Approximate formula:

$$\rho_1 = \left[\rho_c^{1/n} w_n - \frac{1}{n K \gamma} (\Phi_c + \Delta C) \right]^n$$

EXAMPLE:
ELEMENTARY FUNCTIONS

For $n = 1$, $\Omega(r) = \Omega_0/(1 + r^2/A^2)$ and
 $\Delta C = \Phi_c(R_0 = \pi)$ we get:

$$\rho_1(r, z) = \frac{\sin \sqrt{r^2 + z^2}}{\sqrt{r^2 + z^2}} + \frac{1}{2} \frac{\Omega_0^2 A^2 r^2}{1 + \frac{r^2}{A^2}} - \frac{1}{2} \frac{\Omega_0^2 A^2 \pi^2}{1 + \frac{\pi^2}{A^2}}$$