Hoop Conjecture and trapped surfaces in nonspherical massive systems [a]

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Necessary and sufficient conditions for the formation of trapped surfaces in a class of nonspherical systems are proven. Massive singularities, if they exist in the geometries discussed, are always surrounded by apparent horizons. Milder singularities that are strongly nonspherical might be naked. Implications concerning the validity of the hoop conjecture are discussed.

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There is a folk belief that if matter is concentrated in a sufficiently small volume, then the system has to collapse to a black hole. Thorne [1] has formalized this conviction as follows (Hoop Conjecture): black holes form when and only when an Arnowitt-Deser-Misner (ADM) mass gets compacted into a region whose circumference in every direction is $C \leq 4\pi m$. Thus the hoop conjecture contains a strict estimation, as well as a suggestion [called the Compactness Conjecture (CC) hereafter] that the formation of black holes is possible if and only if matter is compressed in all three dimensions. The CC is related to the Trapped Surface (TS) conjecture of Seifert[2] - that massive singularities have to be trapped.

The hoop conjecture understood as a strict criterion is known to fail, at least for long-range fields [3]. The CC however, is supported by the recent analysis of spheroidal geometries made by Shapiro and Teukolsky [4] and (with some reservations) by Bizon and Malec [5], in which the existence of aparent horizons (AH) rather than the event horizon (EH) was studied.

The absence of an AH does not necessarily imply the absence of a global event horizon, although the converse is true (assuming cosmic censorship). Apparent horizons, on the other hand, are much easier to locate than event horizons, whose identification requires the construction of the whole of the space time. Shapiro and Teukolsky's [4] numerical results are consistent even with the numerical part of the hoop conjecture, while in [5] other conditions are formulated with a total rest mass instead of the ADM mass. Also, [5] contains an exact estimation from above for the amount of total rest mass inside a configuration of a fixed largest diameter, from which implicitly follows that strongly eccentric massive oblate spheroids must not be surrounded by trapped surfaces. But, those spheroids cannot form massive singularities, since in a finite volume only a finite amount of matter may be squeezed.

In this paper I probe the Cauchy surface for the appearance of apparent horizons in a class of conformally flat and momentarily static initial data for the Einstein equations. Under these conditions, gravitational waves are absent; this allows us to use the total proper mass as the quasi-local measure of the gravitational energy, as in [5] and [6]. All relevant constants that would appear below are normalized to 1. Let Σ be a time-symmetric and conformally flat Cauchy surface with the asymptotically flat 3-metric g_{ab}. Let S be a closed two-surface of Σ, with the induced metric h_{ab} and the extrinsic curvature p_{ab}. The total mean curvature of S is denoted by $H(S)$:

$$H(S) = \int_S p_{ab} h^{ab} dS. \tag{1}$$

A surface S of nonpositive total mean curvature is called an averaged trapped surface(ATS), as distinct from a trapped surface for which the mean curvature $p_{ab}h^{ab}$ is non-positive pointwise. In physical terms, TS is a closed 2-space with the property that narrow beams of light orthogonal to it at any point decrease in area, at least initially, when propagating outwards; thus the intensity of light increases, when moving outwards. TS's are related to EH's (see, in this connection, an interesting paper by Israel [7]) and black holes, e.g., via the famous Hawking-Penrose singularity theorems [8]. AH (inner or outer) is a boundary (inner or outer) of the region with TS's.

The initial data for the Einstein equations $g_{ab} = f^4 \hat{g}_{ab}$ must satisfy the hamiltonian constraint

$$\hat{\Delta} f = -2\pi \varrho f^5, \tag{2}$$

where ϱ is the matter density, $\hat{g}_{ab}$ is the Euclidean 3-metric and $\hat{\Delta}$ is the laplacian with respect to the flat metric $\hat{g}_{ab}$. Multiplying (2) by f, integrating over the volume $\hat{V}$ inside S and expressing $H(S)$ in terms of the background metric $\hat{g}_{ab}$, one arrives at the following identity:

$$H(S)/8\pi = D(S) - M(S), \tag{3}$$

where

$$2\pi D(S) = \int_V \hat{g}^{\sigma\sigma}(\partial_\sigma f)^2 d\hat{V} + \int_S f^2 \hat{p} d\hat{S}/4, \tag{4}$$

$$H(S) = \int_S [f^2 \hat{p} + 4 f \hat{n}^\sigma \partial_\sigma f] d\hat{S}, \tag{4a}$$

and

$$M(S) = \int_V \varrho dV. \tag{5}$$

$\hat{n}$ is an orthonormal (in $\hat{g}$-metric) vector to the surface S. Here and below all careted quantities refer to the background metric $\hat{g}_{ab}$ while the remaining quantities use the metric g_{ab}. A surface S is called an **equipotential surface** if the conformal factor f is constant on S.

For a convex surface $\hat{S}$ in a flat space the estimation

$$\int_{\hat{S}} \hat{p} d\hat{S} \geq 4\sqrt{\pi \hat{S}} \tag{6}$$

holds[9]. In the formula (6) and below, S might denote either a surface or an area of a surface. Using (6), we obtain for a convex equipotential surface S the estimation

$$D(S) \geq f^2(S) \int_S \hat{p} d\hat{S}/8\pi \geq f^2(S)\sqrt{\hat{S}/\pi}/2 = \sqrt{S/\pi}/2. \tag{7}$$

Thus we may infer the following :

Theorem 1. A necessary condition for an equipotential convex surface S to be an averaged trapped surface (and thus : trapped surface pointwise) is that

$$M(S) \geq \sqrt{S/\pi}/2. \tag{8}$$

This inequality is sharp: in the spherically symmetric case it yields the exact [6,10] estimation

$$M(S) \geq R, \tag{9}$$

where R is the Schwarzschild radius.

Theorem 1 proves a part of the "only if" statement of CC. If one starts with a given amount of matter and compresses it in at least two directions, then the condition (8) will be always met. But the compression in one dimension most likely will not produce a trapped surface, except in the case when a very long and thin body is compacted along its length. Then its surface becomes significantly smaller and the necessary condition could be satisfied.

Let us point out that Theorem 1 does not require that all equipotential surfaces are convex. It is sufficient that only those surfaces are convex, that are suspected to be trapped (e.g., close to the surface of the compacted object). Also, the statement of Th.1 is stable under small deviations either from equipotentiality or convexity.

To proceed further, I will employ a foliation adapted to equipotential surfaces, to have the line element

$$ds^2 = f^4(\sigma)[\hat{g}_{\sigma\sigma}d\sigma^2 + \hat{g}_{ij}dx^i dx^j], i, j = 2, 3. \tag{10}$$

Here $\sigma \geq 0, \sigma$ foliates the level surfaces of the conformal factor f that from now on are assumed to be convex, $\hat{g}_{ab}$ is a flat metric and x^2, x^3 are quasi-angle variables. The covariant vector normal in $\hat{g}-$metric to a surface $\sigma = const$ is given by $\hat{n} = (\hat{n}_\sigma, 0, 0)$, where $\hat{n}_\sigma = \sqrt{\hat{g}_{\sigma\sigma}}$.

For an arbitrary equipotential surface that is ATS but not trapped pointwise, one gets after a simple algebra that

$$\int_S n^\sigma[(n_\sigma p)_{max} - n_\sigma p]dS + 8\pi D(S) \geq 8\pi M(S), \tag{11}$$

(no summation over σ) where $p = h^{ab}p_{ab} = g^{ij}\nabla_i n_j = \nabla^i n_i$ and n is a unit normal in g-metric to S. This formula essentially demonstrates that the larger the nonsphericality [i.e., the larger the integrand of (11)], the more energy that would be packed into the system without creating a trapped surface. Or, in other words, the greater the deviation from spherical symmetry, the more energy that is needed to form the TS, i.e., to reverse the inequality (11). This inequality would also suggest that massive singularities are always trapped if we were able to interpret terms on the left hand side of (11) as related to the volume or a length. The remaining part of this paper is devoted to clarifying this point.

There is a useful identity (proven in [11] but certainly known by mathematicians)

$$\partial_\sigma \int_{\hat{S}} \hat{p} d\hat{S} = 2 \int_{\hat{S}} \hat{K}\hat{n}_\sigma d\hat{S} \equiv 8\pi C(\sigma), \tag{12}$$

where $\hat{S}$ is a smooth surface (that belongs to a given foliation) in 3-dimensional Euclidean space, $\hat{K}$ is its Gauss curvature, and the flat metric of (10) is used.) Differentiation of (4) and use of (12) yield the following identity:

$$\partial_\sigma D(S(\sigma)) = (\partial_\sigma f)^2 \int_{\hat{S}} \hat{n}^\sigma d\hat{S}/(2\pi) + +f(\partial_\sigma f) \int_{\hat{S}} \hat{p} d\hat{S}/(4\pi) + f^2(\sigma)C(\sigma). \tag{13a}$$

Now it is easy to check that

$$D(S(\sigma)) = D(S(\sigma=0)) + \int_0^\sigma \partial_s D(s) ds =$$

$$D(S(0)) + \int_0^\sigma [H(S(s))\partial_\sigma f/(4\pi f) - \int_{\hat{S}} (\partial_\sigma f)^2 \hat{n}^\sigma d\hat{S}/(2\pi) + C(s)f^2(s)]ds \qquad (13)$$

The last line follows from (4), (4a) and (13a).

From (13) it follows that when averaged trapped surfaces are absent inside S (that is, H is positive), then (since $\partial_\sigma f \leq 0$, from the maximum principle)

$$D(S(\sigma)) \leq D(S(0)) + \int_0^\sigma C(s) f^2(s) ds \equiv Rad(\sigma). \qquad (14)$$

From this and the Eq.(3) we conclude that:

Theorem 2. If the content of matter $M(S)$ inside S exceeds the quantity $Rad(S)$, then there must exist averaged trapped surfaces inside S.

By the Gauss-Bonnet theorem

$$Rad(S(\sigma)) \leq \int_0^\sigma f^2(s)\sqrt{sup_{s=const}\hat{g}_{\sigma\sigma}} ds + D(S(0)), \qquad (15)$$

while the quantity D(S(0)) can be bounded from above [11] by $\pi rad(0)/4$, where $rad(0)$ is the radius of the disc on which the conformal factor achieves its maximal value. In the case of spherical symmetry $rad(0)$ vanishes, of course.

The quantity $Rad(S)$ equals to the proper radius in spherical geometries and appears to be bounded from above by the largest proper radius $L(S)$ of the volume enclosed by the surface S in all spheroidal geometries [5] .**I conjecture that this is always true when the equipotential surfaces are convex.**

Theorem 2 is strict, since it gives the same coefficients as in the spherically symmetric case [6].

We will show below that (14) holds for a large class of metrics even in the presence of trapped surfaces. Assume that all equipotential surfaces inside S have the following property [b]:

$$F \equiv (\int_S \hat{p} d\hat{S})^2 - 4\int_S \hat{K}\hat{n}_\sigma d\hat{S} \int_S \hat{n}^\sigma d\hat{S} \geq 0. \qquad (16)$$

A few remarks are in order. The equality in (16) is achieved in the case of spherical symmetry. The examination of foliations by spheroidal surfaces (both prolate and oblate), shows that the left hand side of (16) is the more positive the greater the eccentricity of surfaces. This gives incentive to make my second conjecture that **all convex (and, perhaps, sufficiently smooth) foliations in 3-dimensional Euclidean space satisfy the inequality (16).**

Now we are in a position to prove the following.

Lemma . Let f be a solution of the Lichnerowicz equation (2), with a positive energy density and the boundary condition $f(\infty) = 1$. Let $\hat{S}$ be a level surface of f. Then [c].

$$L(f) \equiv f + 2\int_{\hat{S}} \sqrt{\hat{g}^{\sigma\sigma}}\partial_\sigma f d\hat{S} / \int_{\hat{S}} \hat{p} d\hat{S} \geq 1. \tag{17}$$

Proof.
Differentiate both sides of (17) with respect to σ . Then

$$\partial_\sigma L(f) = F\partial_\sigma f/(\int_{\hat{S}} \hat{p} d\hat{S})^2 + 2\int_{\hat{S}} (\hat{\Delta} f)\hat{n}_\sigma d\hat{S} / \int_{\hat{S}} \hat{p} d\hat{S} \leq$$
$$F\partial_\sigma f/(\int_{\hat{S}} \hat{p} d\hat{S})^2 \leq 0. \tag{18}$$

The first inequality is due to the non-negative energy condition, while the second inequality follows from $\partial_\sigma f \leq 0$, and from (16) . Thus $L(f)$ decreases: but it is equal to 1 at spatial infinity, so the lemma is proved.

Having (17) we may replace the last line of (13) by another expression:

$$D(S(\sigma)) = D(S(0)) + \int_0^\sigma \ dsL(f)\partial_\sigma f \int_{\hat{S}} \hat{p} d\hat{S}/(4\pi) + \int_0^\sigma f^2(s)C(s)ds$$
$$\leq Rad(\sigma). \tag{19}$$

The last inequality results from (17). As was already pointed out, the estimation (19) is independent of the existence of trapped surfaces.Let us remark also that in fact both of the conditions (16) and (17) are probably too strong - (17) may be true even for surfaces that violate (16) and (19) would hold even if (17) were not satisfied.Together with the remarks following the estimation (16), these facts allow one to believe in validity of (19) in the whole class of convex geometries (i.e., geometries with equipotential surfaces being convex).

It is an important fact to stress out that the inequalities (17) and (19) hold for moving matter. In contrast to the preceding results, these two estimations remain true also for nonsymmetric in time initial data (on the maximal Cauchy slice) for the Einstein equations. We have used only the Lichnerowicz equation (2) and in the case of nonsymmetric in time initial data the only change that occurs is the appearence of a new negative term on the right hand side of (2) related to the extrinsic curvature of the Cauchy surface. Thus the maximum principle still can be applied and the whole idea of the proofs remains intact.

The fact that (19) actually does not depend on whether there are trapped surfaces or not allows us to reformulate Theorem 2 as follows.

Theorem 2'. If the content of matter $M(S)$ inside S exceeds $Rad(S)$ (or $L(S)$, the largest proper radius of a volume enclosed by S, if the first conjecture is accepted) then S itself is an ATS.

Now we may discuss the sufficient conditions for the formation of pointwise trapped surfaces. If $M(S)$ exceeds $Rad(S)$ then S is an ATS, according to theorem 2'. If S is to become pointwise trapped, then from inequality (11) the amount of matter inside a volume of a fixed $Rad(S)$ should be as great as needed to balance the deviation from nonsphericality represented by the integrand on the left hand side of (11) and to reverse this inequality. In short terms, we proved the following:

Proposition If at S the following inequality is satisfied

$$M(S) \geq Rad(S) + \int_S [n^\sigma (n_\sigma p)_{max} - p] dS,$$

(no summation over σ) then S is pointwise trapped.

Thus, if the content of matter inside S is made infinitely large for fixed $Rad(S)$, then S will always become a trapped surface. In this sense massive singularities on a momentarily static Cauchy surface are always trapped, according to Seifert's conjecture [2]. The point is, however, that in the class of geometries discussed the ratio $M(S)/Rad(S)$ is bounded from above.

Theorem 3. If the initial (nonsymmetric in time) data of Einstein's equations are conformally flat on the maximal Cauchy slice, with the level surfaces of the conformal factor being convex inside S and satisfying the condition (16), then

$$M(S) \leq Rad(S) + \int_{\hat{S}} f^2 \hat{p} d\hat{S}$$

$$\leq 2Rad(S). \tag{20}$$

Proof.

$$M(S) \leq - \int_{\hat{V}} f \hat{\Delta} f d\hat{V}/(2\pi)$$

$$= D(S) + \int_{\hat{S}} f^2 \hat{p} d\hat{S}/(8\pi) - Lf \int_{\hat{S}} \hat{p} d\hat{S}/(4\pi) \leq$$

$$Rad(S) + \int_{\hat{S}} f^2 \hat{p} d\hat{S}/(8\pi)$$

$$\leq 2Rad(S). \tag{21}$$

The first line follows directly from the definition of $M(S)$ and Eq. (2) (which for nonsymmetric in time initial data contains also a negative term with the extrinsic curvature) and the last two inequalities follow from (4), (14) and (17).

Theorem 3 is exact, since it reproduces the hitherto known bounds in spherically symmetric [12,13] and spheroidal geometries [5],

$$M(S) \leq 2Rad(S), \tag{22}$$

that are saturated by massive shells. In those cases $Rad(S)$ is either equal or smaller than the largest proper radius of the configuration.

To summarize the whole discussion: If a maximal slice with a conformally flat and convex (in the sense defined above) 3-geometry is considered, then the volume $V(S)$ of a fixed size $Rad(S)$ can contain the amount of matter $M(V)$ not greater than $2Rad(S)$. If momentarily static initial data are considered, then the necessary condition for S to be trapped is that the total rest mass satisfies the condition (8), that is rougly speaking, that the area of S is very small. This result essentially proves the "only if"part of the hoop conjecture (or, strictly speaking, of CC). Theorem 2' gives a sufficient condition for a surface to be trapped in the average, but if the geometry is strongly nonspherical, then an equipotential surface may not be trapped, regardless of the amount of matter inside it. In this sense the geometry of the Cauchy surface might be "singular", but in fact this singularity is very mild, because of the universal bound on the total rest mass . Relating that to the hoop conjecture, we would say that it essentially proves the "if" part of the Compactness Conjecture, with this restriction that although trapped surfaces will form when matter is compressed in all dimensions, in addition, *the compression should be sufficiently "round"* . This reservation is related to the existence of the aforementioned bound. All the results are valid for compact and noncompact distributions of matter and for geometries that are convex (and satisfy the condition (16)) or almost convex (in a sense that could be precisely defined).

It is necessary to point out that, since we work on the Cauchy surface, there is no direct connection between the present investigation and that of Shapiro and Teukolsky[4]. They analyzed the evolution of a system that was initially spheroidal and at rest; the problem is that in later evolution it would stop to belong to the class of geometries discussed here. The numerical data of [4] are not sufficient to make comparisons with statements of this paper.

To conclude, let me remark that the criteria proven in this paper constitute isoperimetric inequalities[9] in asymptotically and conformally flat Riemannian manifolds. (Gibbons was the first to connect isoperimetric inequalities with the qualitative description of black holes [14].) In the language of isoperimetric inequalities, estimates are given above of the total mean curvature of a two-dimensional convex surface S in terms of the total curvature inside S, its area and/or the largest diameter of the surface. Most of the results are not known in the mathematical literature, to my knowledge.

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[d] Comments

[b] This inequality can be actually proven for the inverse mean curvature foliations (i. e., satisfying the Geroch condition).

[c] The inequality in question was essentially proven by Szëgo, in the context of the relation between the electrostatic capacity and the total mean curvature (E. Flanagan, private communication). A detailed proof, adapted to the above case, is given in Acta Phys. Pol. **B23**, 123(1992).

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