Comment on the Herzlich's proof of the Penrose inequality [a]

Edward Malec

Jagiellonian University, Institute of Physics, 30-59 Kraków, Reymonta 4, Poland

Krzysztof Roszkowski

Jagiellonian University, Institute of Physics, 30-59 Kraków, Reymonta 4, Poland

(August 2, 2000)

Abstract

Recently Herzlich proved a Penrose-like inequality with a coefficient being a kind of a Sobolev constant. We show that this constant tends to zero for charged black holes approaching maximal Reissner - Nordström solutions. The method proposed by Herzlich is not appropriate for charged matter with nonzero global charge.

[a] Acta Phys. Pol. **29**, 1975(1998)

One of the most interesting problems of General Relativity was formulated in 1973 by R. Penrose [1] in his attempt to specify conditions under which a cosmic censorship hypothesis can be broken.

Specified entirely in terms of initial data, the **Penrose Conjecture** says the following.

Let(M,g) be a 3-dimensional asymptotically flat Riemannian manifold with a compact, connected, minimal and stable (inner) boundary ∂M which is a topological 2-sphere. Suppose also that the scalar curvature of (M,g) is nonnegative. Then its mass, if defined, satisfies:

$$m \geq \sqrt{\frac{Area(\partial M)}{16\pi}}$$

and equality is achieved if and only if (M,g) is a spacelike Schwarzschild metric.

A proof of this inequality is known only in special cases ([2], [3]). In 1997 Marc Herzlich presented a refreshingly new approach to that problem in the case of momentarily static initial data. His main theorem cited below in extenso [4] shows a Penrose - like inequality with a functional coefficient $\tau = \frac{2\sigma}{1+\sigma}$.

Let(M,g) be a 3-dimensional asymptotically flat Riemannian manifold with a compact, connected, minimal and stable (inner) boundary ∂M which is a topological 2-sphere. Suppose also that the scalar curvature of (M,g) is nonnegative. Then its mass, if defined, satisfies:

$$m \geq \frac{2\sigma}{1+\sigma}\sqrt{\frac{Area(\partial M)}{16\pi}},$$

where σ is a dimensionless quantity defined as

$$\sigma = \sqrt{\frac{Area(\partial M)}{\pi}} \inf_{\phi \in C_c^\infty, \phi \neq 0} \frac{||d\phi||^2_{L^2(M)}}{||\phi||^2_{L^2(\partial M)}}.$$

Moreover, equality is achieved if and only if (M,g) is a spacelike Schwarzschild metric of mass $\frac{1}{4}\sqrt{Area(\partial M)/\pi}$.

In what follows we show that τ can be arbitrarily small in the case of spherical charged black holes that are almost maximal.

Let us assume a spherically symmetric spacetime and choose isotropic coordinates. Then spatial part of the line element reads

$$ds^2 = f^4(r)(dr^2 + r^2 d\Omega^2), \tag{1}$$

where $d\Omega^2$ is the line element on a unit sphere and $0 \leq r < \infty$.

It is convenient to normalize the function ϕ so that ϕ is equal to 1 at the inner boundary. Restriction to the set of spherically symmetric minimizers yields

$$\sigma = 2 \inf_{\phi:\phi(r_0)=1} \frac{\int_{r_0}^{\infty} (\partial_r \phi)^2 f^2 r^2 dr}{r_0 f^2(r_0)}. \tag{2}$$

A minimizing funtion ϕ satisfies a harmonic equation which reads, assuming spherical symmetry,

$$\frac{1}{r^2 f^6} \frac{d}{dr}(r^2 f^2 \phi') = 0. \tag{3}$$

Therefore $\phi' = \frac{C}{r^2 f^2}$ and

$$\phi(r) = 1 + \int_{r_0}^{r} \frac{C}{f^2(r') r'^2} dr'. \tag{4}$$

The condition that ϕ vanishes at infinity leads to the integration constant C

$$C = \frac{-1}{\int_{r_0}^{\infty} ds \frac{1}{s^2 f^2(s)}}. \tag{5}$$

The integrand of (2) can be written as

$$\begin{aligned} &\int_{r_0}^{\infty} \phi' \left(\phi' f^2 r^2\right) dr = \phi \left(\phi' f^2 r^2\right)|_{r_0}^{\infty} - \int_{r_0}^{\infty} \phi' \frac{d}{dr} \left(\phi' f^2 r^2\right) dr = \\ &-(\phi \left(\phi' f^2 r^2\right))(r_0) = |C|. \end{aligned} \tag{6}$$

From this we finally arrive at

$$\sigma = \frac{2}{f^2(r_0) \int_{r_0}^{\infty} ds \frac{1}{s^2 f^2(s)}}. \tag{7}$$

Assume a fixed momentarily static Cauchy slice. One can smoothly join an interior geometry of a charged matter region with the external (electrovacuum) geometry as given by the conformal factor f,

$$f = \sqrt{1 + \frac{m}{r} + \frac{m^2 - q^2}{4r^2}}. \tag{8}$$

Explicit integration of the integral of (5) yields, in electrovacuum,

$$\int_{r_0}^{\infty} ds \frac{1}{s^2 f^2(s)} = \frac{1}{|q|} \ln(1 + \frac{|q|}{r_0 + m/2 - |q|/2}); \tag{9}$$

therefore

$$\sigma = \frac{2m}{\ln(1 + \frac{|q|}{r_0 + m/2 - |q|/2})(r_0 + m + \frac{m^2 - q^2}{4r_0})}. \tag{10}$$

If r_0 is a coordinate radius of a minimal surface, then

$$\frac{d}{dr}\left(r^2 f^4\right) = 0$$

In electrovacuum the last equation yields the radius r_0 of a minimal sphere as a function of m and q,

$$r_0 = \sqrt{\frac{m^2 - q^2}{4}}. \tag{11}$$

That implies that at the specified radius

$$f(r_0) = \sqrt{2 + \frac{2m}{\sqrt{m^2 - q^2}}}. \tag{12}$$

We may choose parameters so that $m^2 - q^2$ is arbitrarily small. Then the denominator in (10) becomes arbitrarily large and both σ and τ may be as small as one wishes [5]. In this limit the result of Herzlich reduces merely to the positivity of the asymptotic mass m. We would like to point out that this does not imply that the Penrose inequality is not valid for the charged matter (in fact it holds true), but that the method discussed above is not appropriate.

It is interesting to note that charged matter again and again causes troubles in various conjectures. Bonnor has shown [6] that charged matter falsifies the standard form of the hoop conjecture [7]. Then a compactness criterion of Yau and Schoen [8] for the formation of horizons has been shown to fail [9]. An isoperimetric inequality proposed by Gibbons [2] can be broken by distributions with a bulk of charge outside of a horizon [3].

Acknowledgement One of the authors (EM) gratefully acknowledges discussions with Marc Herzlich. This paper is in part supported by the KBN grant 2 PO3B 090 08.

REFERENCES

[1] R. Penrose, Naked Singularities Ann. N. Y. Acad. Sci. **224**, 125-134 (1973)

[2] G. Gibbons, in *Global Riemannian Geometry* ed. N. J. Willmore and N. J. Hitchin, Cambridge University Press (1984).

[3] E. Malec and N. O' Murchadha, *Phys. Rev.* **D49**, 6931(1994); Appendix in Phys. Rev. **D54**, 4792(1996)

[4] M. Herzlich, Commun. Math. Phys. **188**, 121-133 (1997).

[5] Niall O'Murchadha and Marc Herzlich informed us recently that Paul Tod arrived at the same conclusion.

[6] W. B. Bonnor, *Phys.Lett.* **A99**, 424(1983)

[7] C. W. Misner, K. Thorne, J. Wheeler, Gravitation (Freeman, San Francisco 1973), p.867.

[8] R. Schoen, S. T. Yau, *Commun.Math.Phys.* **90**, 575(1983)

[9] P. Bizoń, E. Malec and N. O'Murchadha, *Class. Quantum Grav.* **6**, 961(1989).