Quantum Mechanics III, set 7.

Ex. 1. Exercise to calculate normal ordering of the creation and annihilation operators. For the bosonic case with a single mode we have operators $a,\ a^\dagger,\ [a,a^\dagger]=1$. Show that

a)
$$e^{\lambda a^\dagger a} =: e^{(e^\lambda - 1)a^\dagger a} :$$

b)
$$e^{\lambda(a+a^\dagger)} = e^{\lambda^2/2} : e^{\lambda(a+a^\dagger)} :$$

Ex. 2.

a) Show that for the charged K-G field the Hamiltonian has a form

$$H = \int (dp) E_p \left(A^\dagger(\vec{p}) A(\vec{p}) + B^\dagger(\vec{p}) B(\vec{p}) \right), \quad (dp) \equiv \frac{d^3p}{(2\pi\hbar)^3 2E_p}.$$

Find a form of the operators $\Phi_H(x)$ and $\Phi_H^\dagger(x)$ for the charged Klein-Gordon field in Heisenberg picture.

a) Show that for the neutral K-G field the Hamiltonian has a form

$$H = \int (dp) E_p \left(a^\dagger(\vec{p}) a(\vec{p}) \right).$$

Find a form of the operator $\Psi_H(x)$ for the neutral Klein-Gordon field in Heisenberg picture.

Ex. 3. Calculate the wave functions of the one-particle "particle" and "anti-particle" states of the charged K-G field:

$$\phi^p_{\vec{p}}(x) = \langle 0|\Phi_H(x) A^\dagger(\vec{p})|0\rangle, \quad \phi^a_{\vec{p}}(x) = \langle 0|\Phi^\dagger_H(x) B^\dagger(\vec{p})|0\rangle.$$

Show that they satisfy the K-G equation.
Calculate the wave function of the one-particle state of the neutral K-G field:

$$\psi^p_{\vec{p}}(x) = \langle 0|\Psi_H(x)a^\dagger(\vec{p})|0\rangle.$$

Ex. 4. For a charged (complex) K-G field find a form of the operator cooresponding to the "conserved" density of charge

$$Q_{KG} =: i\hbar \int d^3x \left(\phi^\dagger_H(\vec{x},t)(\partial_t\phi_H(\vec{x},t)) - (\partial_t\phi^\dagger_H(\vec{x},t))\phi_H(\vec{x},t)\right) :$$

Check the role played by the normal ordering in this expression.
Show that Q_{KG} commutes with the Hamiltonian.

Ex. 5. Interacting K-G field. Determine the classical Euler-Lagrange equations for the complex field described by Lagrangian densities:

a)

$$\mathcal{L} = \hbar^2c^2g^{\mu\nu}(\partial_\mu\Phi(x)^*)(\partial_\nu\Phi(x)) - m^2c^4\Phi^*\Phi - j(x)\Phi^* - j^*(x)\Phi.$$

Here $j(x)$ and $j^*(x)$ are external functions.

b)

$$\mathcal{L} = \hbar^2c^2g^{\mu\nu}(\partial_\mu\Phi(x)^*)(\partial_\nu\Phi(x)) - m^2c^4\Phi^*\Phi - \Phi^*W(x)\Phi.$$

Here $W(x)$ is an external function.

Ex. 6. Complex K-G field. Calculate

$$T\left(\Phi(x)\Phi(y)\right) - :\Phi(x)\Phi(y): \quad T\left(\Phi(x)\Phi^\dagger(y)\right) - :\Phi(x)\Phi^\dagger(y):$$

Real K-G field. Calculate

$$T\left(\phi(x)\phi(y)\right) - :\phi(x)\phi(y):$$

T means a time-ordered product.