Quantum Mechanics III. Set 6.

Ex. 1. Repeat the derivation of the Euler-Lagrange equations. The action S is given by

$$S = \int_{t_1}^{t_2} dt L$$

where the Lagrangian L

$$L = \int d^3x \mathcal{L}\left(\Psi_\sigma(\vec{x},t), \frac{\partial \Psi_\sigma(\vec{x},t)}{\partial x^\mu}\right), \quad \mu = 0,1,2,3.$$

$\mathcal{L}$ is the Lagrangian *density* and σ numbers the *fields*.
Show that from the variational principle $\delta S = 0$ under conditions $\delta\Psi_\sigma(\vec{x},t_1) = \delta\Psi_\sigma(\vec{x},t_2) = 0$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi_\sigma(\vec{x},t))} - \frac{\partial \mathcal{L}}{\partial \Psi_\sigma(\vec{x},t)} = 0.$$

What are these equations if x^0 is expressed by time t and spatial coordinates $\vec{x}$? We assume that fields $\Psi_\sigma(\vec{x},t)$ vanish sufficiently fast for all t when $\vec{x} \to \infty$.

Ex. 2. Show that if a Lagrangian density has a form

$$\mathcal{L} = -\frac{\hbar^2}{2m}\left(\partial_i \Psi^*(\vec{x},t)\right)\left(\partial_i \Psi(\vec{x},t)\right) - \Psi^* V(\vec{x})\Psi + \frac{i\hbar}{2}\left(\Psi^*\frac{\partial \Psi}{\partial t} - \frac{\partial \Psi^*}{\partial t}\Psi\right)$$

and assuming fields Ψ and Ψ^* are independent, the E-L equations are identical to the one-particle Schródinger equation (both for Ψ and Ψ^*).
Check that the same equations are obtained if the term with time derivatives is replaced by

$$\text{a) } i\hbar\left(\Psi^*\frac{\partial \Psi}{\partial t}\right), \quad \text{b) } -i\hbar\left(\frac{\partial \Psi^*}{\partial t}\Psi\right).$$

Why?

Ex. 3. Assume that the Lagrangian density has a form b) from ex. 2. Perform the canonical quantization in this case (find the canonical momentum conjugate to $\Psi^*(\vec{x}, t)$, determine the canonical commutation relations, find the Hamiltonian and show that result is the same as for the case presented on the lecture).

Ex. 4. Assume that the Lagrangian density has a form

$$\mathcal{L} = \hbar^2 c^2 g^{\mu\nu} (\partial_\mu \phi^*(x))(\partial_\nu \phi(x)) - m^2 c^4 \phi^*(x)\phi(x)$$

with $x = \{x^0, x^2, x^2, x^3\}$, $x^0 = ct$.

- Determine the E-L equations assuimng that fields ϕ^* and ϕ are independent.
- Rewrite the Lagrangian explicitly in time and space derivatives of fields. Find canonical momenta $\pi(x)$ and $\pi^*(x)$ conjugate respectively to $\phi(x)$ and $\phi^*(x)$.
- Derive a form of the Hamiltonian.

Hint: Use the concept of a variational derivative $\delta L/\delta(\partial_t \phi(x))$.

Ex. 5. Assume that the Lagrangian density has a form

$$\mathcal{L} = \frac{\hbar^2 c^2}{2} g^{\mu\nu} (\partial_\mu \phi(x))(\partial_\nu \phi(x)) - \frac{m^2 c^4}{2} \phi(x)^2$$

with $x = \{x^0, x^2, x^2, x^3\}$, $x^0 = ct$ and $\phi(x)$ a real scalar field.

- Determine the E-L equations.
- Rewrite the Lagrangian explicitly in time and space derivatives of fields. Assume that field $\phi(x)$ is periodic in a spatial box $L \times L \times L$. Decompose field $\phi(x)$ as a combination of momentum eigenstates in a periodic box

  $$\phi(x) = \sum_{\vec{n}} c_{\vec{n}}(t) \psi_{\vec{n}}(\vec{x}), \quad \psi_{\vec{n}}(\vec{x}) = \frac{1}{L^{3/2}} e^{2\pi i \vec{n}\vec{x}/L}.$$

  What relations between coefficients $c_{\vec{n}}(t)$ follow from the fact that field $\phi(x)$ is real?
- Derive a form of the Hamiltonian.
- Find independent degrees of freedom and quantize the system.