Quantum Mechanics III. Set 5.

Ex. 1. We assume that the one-particle non-relativistic hamiltonian H has discrete eigenstates $|n\rangle$

$$H|n\rangle = E_n|n\rangle, \quad \langle n|m\rangle = \delta_{nm}.$$

A set of N identical, indistinguishable particles of this type is described by a hamiltonian $\mathcal{H}$

$$\mathcal{H} = \sum_{j=1}^{N} H_j,$$

where H_j is a hamiltonian for the particle j. The eigenstates of $\mathcal{H}$ in the Fock space have a form $|N_1, N_2, \dots N_k, \dots\rangle$, where $N_k = 0, 1, \dots$ is a number of particles in the state k. The *vacuum* state $|0\rangle \equiv |0, 0, 0, \dots, 0, \dots\rangle$ is defined as a state without particles with a norm $\langle 0|0\rangle = 1$.
A one-particle state in the eigenstate $|n\rangle \equiv |0, 0, \dots, 1_n, 0 \dots\rangle$ can be obtained acting with a *creation* aperator $a_n^\dagger|0\rangle$. Defining a conjugate operator of *annihilation* a_n we assume that

$$a_n|0\rangle = 0.$$

Operators of creation and annihilation of one-particle states satisfy

$$[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0, \quad [a_i, a_j^\dagger] = \delta_{ij}, \quad \text{for bosons},$$

$$\{a_i, a_j\} = \{a_i^\dagger, a_j^\dagger\} = 0, \quad \{a_i, a_j^\dagger\} = \delta_{ij}, \quad \text{for fermions}.$$

a. How can we construct an arbitrary state $|N_1, N_2, \dots N_k, \dots\rangle$ acting with creation operators on a vacuum state (both bosonic and fermionic case)? States should be normalized to unity.

b. Show that both for bosons and fermions the hamiltonian is

$$\mathcal{H} = \sum_k E_k a_k^\dagger a_k.$$

c. Above definitions are for the Schrödinger picture, where the states of the system depend on time and satisfy the Schrödinger equation. In the Heisenberg picture states are time independent, but operators depend on time. Find the form of creation and annihilation operators in the Heisenberg picture (bosons and fermions).

Ex. 2. The wave function of a one-particle eigenstate $|n\rangle$ is defined by

$$\Psi_n(\vec{x}) = \langle \vec{x}|n\rangle.$$

In the Schrödinger picture we define a field operator

$$\Phi(\vec{x}) = \sum_m \Psi_m(\vec{x}) a_n.$$

a. Show that

$$\Psi_n(\vec{x}) = \langle 0|\Phi(\vec{x})|0, 0, \ldots, 1_n, 0, \ldots\rangle = \langle 0|\Phi(\vec{x})a_n^\dagger|0\rangle.$$

b. The operator conjugate to $\Phi(\vec{x})$ is

$$\Phi^\dagger(\vec{x}) = \sum_n \Psi_n^*(\vec{x}) a_n^\dagger.$$

This operator acting on a vacuum state generates a one-patrticle state $|\vec{x}\rangle$ localized in the point $\vec{x}$. Show that

$$\text{for bosons} \quad [\Phi(\vec{x}), \Phi(\vec{x}')] = 0, \quad [\Phi(\vec{x}), \Phi^\dagger(\vec{x}')] = \delta^{(3)}(\vec{x} - \vec{x}'),$$

$$\text{for fermions} \quad \{\Phi(\vec{x}), \Phi(\vec{x}')\} = 0, \quad \{\Phi(\vec{x}), \Phi^\dagger(\vec{x}')\} = \delta^{(3)}(\vec{x} - \vec{x}').$$

c. Show that in both cases $\langle \vec{x}|\vec{y}\rangle = \delta^{(3)}(\vec{x} - \vec{y})$.

d. Assuming that in a position representation the one-particle hamiltonian has a form

$$H = -\frac{\hbar^2}{2m} \triangle_x + V(\vec{x})$$

show that

$$\mathcal{H} = \int d^3x \Phi^\dagger(\vec{x}) \left(-\frac{\hbar^2}{2m} \triangle_x + V(\vec{x}) \right) \Phi(\vec{x}).$$

e. Find a form of $\Phi_H(\vec{x;t})$ in the Heisenberg picture.

f. Show that the operator in the Heisenberg picture $\Phi_H(\vec{x;t})$ satisfies the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\Phi_H(\vec{x;t}) = \left(-\frac{\hbar^2}{2m}\triangle_x + V(\vec{x})\right)\Phi_H(\vec{x;t}).$$

Ex. 3. Find a form of a two-particle wave function (bosons and fermions)

$$\Psi_{n_1,n_2}(\vec{x_1},\vec{x_2}) = \langle 0|\Phi(\vec{x_1})\Phi(\vec{x_2})a^\dagger_{n_1}a^\dagger_{n_2}|0\rangle.$$

Check the normalization.

Ex. 4. Particular case: a system contains *free* particles in a cubic box with the edge L (assuming periodic boundary conditions). Write explicit form of the field operator $\Phi(\vec{x})$ in this case. Check what happens in the limit $L\to\infty$, when the dicrete sum over the eigenstates should be replaced by the integral. How should we define the one-particle creation and annihilation operators and their (anti)commutation rules?
Hint: We aim at a new normalization of the one-particle states $|\vec{p}\rangle$ of a form

$$\langle\vec{q}|\vec{p}\rangle = (2\pi\hbar)^3\delta^3(\vec{q}-\vec{p}).$$

Why?