Quantum Mechanics III, set 4.

Ex. 1. Show that if the spinors $w^r(\vec{p})$ satisfy the normalization $w^r(\epsilon_r\vec{p})^\dagger w^{r'}(\epsilon'_r\vec{p}) = 2E_p\delta_{rr'}$ it follows that $\bar{w}^r(\vec{p})w^{r'}(\vec{p}) = 2\epsilon_r mc^2\delta_{rr'}$. Here

$$\bar{w}^r(\vec{p}) = w^r(\vec{p})^\dagger\gamma^0.$$

Calculate quantities

$$a_r^\mu = \bar{w}^r(\vec{p})\gamma^\mu w^r(\vec{p}), \quad b_r^{\mu\nu} = \bar{w}^r(\vec{p})\sigma^{\mu\nu} w^r(\vec{p}), \quad c_r = \bar{w}^r(\vec{p})\gamma_5 w^r(\vec{p}).$$

Ex.2. Show that if we boost the solution of the Dirac equation

$$\Psi^r_{\vec{p}}(x) = e^{-\frac{i}{\hbar}\epsilon_r mc^2 t} w^r(\vec{p}=0)$$

to a reference frame x' where the particle has momentum $\vec{p}$ we obtain

$$\Psi^r_{\vec{p}}(x) \to e^{-\frac{i}{\hbar}\epsilon_r p_\mu x'^\mu} w^r(\vec{p}).$$

Determine the explicit form of the boost transformation.
Hint We need a boost in a direction $\vec{n} = \vec{p}/|\vec{p}|$ with a boost parameter which transforms mc^2 to E_p.

Ex.3. The partity transformation P transforms coordinates $x = \{x^0, x^1, x^2, x^3\}$ to $\tilde{x} = \{x^0, -x^1, -x^2, -x^3\}$. Find a form of the transformation U_P such that

$$\tilde{\Psi}(\tilde{x}) = U_P\Psi(x)$$

which preserves a form of the Dirac equation

$$\left(i\hbar c\gamma^\mu\partial_\mu - mc^2\right)\Psi(x) = 0.$$

Determine how this transformation changes special solutions

$$\Psi^r_{\vec{p}}(x) = e^{-\frac{i}{\hbar}\epsilon_r p_\mu x^\mu} w^r(\vec{p}).$$

Ex.4. The charge conjugation transformation C transforms a spinor $\Psi(x)$ to $\Psi_C(x) = U_C \Psi^*(x)$. Find a form of the transformation U_C which preserves a form of the Dirac equation. Determine how this transformation changes special solutions

$$\Psi^r_{\vec{p}}(x) = e^{-\frac{i}{\hbar}\epsilon_r p_\mu x^\mu} w^r(\vec{p}).$$

Note: Here $*$ means complex conjugation and not Hermitian conjugation!

Ex.5. The time inversion transformation T transforms coordinates $x = \{x^0, x^1, x^2, x^3\}$ to $x' = \{-x^0, x^1, x^2, x^3\}$. Find a form of the transformation U_T such that

$$\Psi_T(x') = U_T \Psi^*(x)$$

which preserves a form of the Dirac equation

$$\left(i\hbar c\gamma^\mu \partial_\mu - mc^2\right)\Psi(x) = 0.$$

Determine how this transformation changes special solutions

$$\Psi^r_{\vec{p}}(x) = e^{-\frac{i}{\hbar}\epsilon_r p_\mu x^\mu} w^r(\vec{p}).$$

Ex.6. We define matrices

$$\Lambda_\pm(p) = \frac{\pm c\gamma^\mu p_\mu + mc^2)}{2mc^2}.$$

Show that these matrices are projections in the space of spinors, i.e.

$$\Lambda_+^2 = \Lambda_+, \quad \Lambda_-^2 = \Lambda_-, \quad \Lambda_= \cdot \Lambda_- = 0.$$

Check the result of $\Lambda_\pm w^r(\vec{p})$.
Show that

$$2mc^2\Lambda_+(p) = \sum_{r=1,2} w^r(\vec{p})\bar{w}^r(\vec{p}) = \sum_{\pm s} u(p,s)\bar{u}(p,s),$$

$$2mc^2\Lambda_-(p) = -\sum_{r=3,4} w^r(\vec{p})\bar{w}^r(\vec{p}) = -\sum_{\pm s} v(p,s)\bar{v}(p,s).$$

We define matrices

$$\Sigma(\pm s) = \frac{\pm\gamma_5\gamma_\mu s^\mu + 1}{2}.$$

Check that these matrices commute with $\Lambda_\pm(p)$ and express the products $2mc^2\Lambda_\pm(p)\Sigma(\pm s)$ through direct products $u(p,s)\bar{u}(p,s)$ and $v(p,s)\bar{v}(p,s)$.