Quantum Mechanics III, set 3.

Ex. 1. Repeat the properties of the Pauli matrices $\sigma_i,\ i = 1, 2, 3$:

$$\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k, \quad (\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b} + i\vec{\sigma}\cdot(\vec{a}\times\vec{b})$$

Ex. 2. Start with the Dirac equation in the form:

$$i\hbar\partial_t\Psi = \left(-ic\hbar(\vec{\alpha}\cdot\vec{\nabla}) + \beta mc^2 + V(r)\right)\Psi$$

Show that

a) This equation can be interpreted as describing interaction of the electron with the electromagnetic field ($\partial_\mu \to D_\mu$) in the case $\vec{A}(x) = 0,\ V(r) = eA_0(r)$

b) Show that the equation is invariant under $\vec{x} \to -\vec{x}$ and $\Psi(\vec{r}, t) \to \beta\Psi(-\vec{r}, t)$.

c) Rewrite the equation as an equation for a stationary state and show that the Hamiltonian commutes with the rotation operators $J_i = L_i + S_i$, where

$$S_i = \frac{\hbar}{2}\Sigma_i, \quad \Sigma_i = \left(\begin{array}{c|c} \sigma_i & 0 \\ \hline 0 & \sigma_i \end{array}\right)$$

d) Using the operators $2\times 2:\ j_i = L_i + \hbar\sigma_i/2$ construct the eigenstates $\vec{j}^2,\ \vec{L}^2,\ j_3$ in the form

$$\Omega_{jlm} = \begin{pmatrix} \alpha_1 Y_{lm-1/2}(\theta,\phi) \\ \alpha_2 Y_{lm+1/2}(\theta,\phi) \end{pmatrix}$$

find the eigenvalues α_1 and α_2. What is the eigenvalue of j_3? Show that eigenstates with a given j can be constructed only from l or l' with $l + l' = 2j$, i.e. $l, l' = j \pm 1/2$.

e) Find a form of the Dirac equation in the parametrization

$$\Psi(\vec{r},t) = e^{-\frac{i}{\hbar}Et} \begin{pmatrix} f(r)\Omega_{jlm} \\ g(r)\Omega_{jl'm} \end{pmatrix}$$

where $l' = 2j - l$ and determine the set of equations for $f(r)$ and $g(r)$.

Hint: Assume a simplified relation (without referring to the standard phase factor convention:)

$$\frac{\vec{\sigma}\cdot\vec{r}}{r}\Omega_{jlm} = \Omega_{jl'm}, \quad \frac{\vec{\sigma}\cdot\vec{r}}{r}\Omega_{jl'm} = \Omega_{jlm}$$

where $l + l' = 2j$.

Ex. 3. Using the reduced form of the Dirac equation analyze the eigenvalue spectrum of the "hydrogen-like" atom $V(r) = -Ze^2/r$:

a) Rewrite the Dirac equation as a set of two equations satisfied by functions $f(r)$ and $g(r)$ at fixed values of quantum numbers $j, m, l, l' = 2j - l$.

b) Check the asymptotic behavior at $r \to \infty$. What restriction should be satisfied by the localized eigenstate? Show that in this limit $f(r)$ and $g(r)$ should behave as e^{-kr}. Calculate k. Check the asymptotics $r \to 0$ (behavior $\propto r^\alpha$).

c) Find a form of the reduced eqation (eliminating the asymptotic behavior at $r \to \infty$) for $F(r)$ i $G(r)$ (where $f(r) = F(r)e^{-kr}$, similarly $G(r)$).

d) Determine the recurrence equation for coefficients of the power expansion

$$F(r) = \sum_n f_n r^{n+\alpha} \text{ and } G(r) = \sum_n g_n r^{n+\alpha}$$

as a two-component matrix equation

$$\begin{pmatrix} f_{n+1} \\ g_{n+1} \end{pmatrix} = A_n \begin{pmatrix} f_n \\ g_n \end{pmatrix}$$

Find eigenvalues A_n and show that this equation permits a determination of the spectrum of localized states.