

## Quantum Mechanics III, set 2.

**Ex. 1.** Consider special solutions of Klein-Gordon equation

$$(\hbar^2 g^{\mu\nu} \partial_\mu \partial_\nu + m^2 c^2) \Psi(\vec{x}; t) = 0,$$

in a form

$$\Psi_{\vec{p}}^\pm(\vec{x}; t) = e^{\mp \frac{i}{\hbar} E_p t + \frac{i}{\hbar} \vec{p} \vec{x}}, \quad E_p = +\sqrt{\vec{p}^2 c^2 + m^2 c^4}.$$

Check the norm of these states, i.e. calculate

$$\int d^3 x i \hbar \left( \Psi_{\vec{q}}^\pm *(\vec{x}; t) (\partial_t \Psi_{\vec{p}}^\pm(\vec{x}; t)) - (\partial_t \Psi_{\vec{q}}^\pm *(\vec{x}; t)) \Psi_{\vec{p}}^\pm(\vec{x}; t) \right).$$

- How this norm transforms under orthochronous Lorentz transformations?
- What are the transformation rules for:  $d^3 p$ ,  $d^3 p / (2E_p)$ ?

**Ex. 2.** We introduce interaction to the K-G equation:

$$(\hbar^2 g^{\mu\nu} \partial_\mu \partial_\nu + m^2 c^2 + W(\vec{x}, t)) \Psi(\vec{x}; t) = 0,$$

where  $W(\vec{x}, t)$  is a given function. Let us assume a particular form of this function

$$W(\vec{x}; t) = -\frac{C}{r}$$

(which means it is time independent and spherically symmetric, with some constant  $C$ ). We are looking for special solutions of this equation in a form

$$\Psi_E(\vec{x}; t) = e^{-\frac{i}{\hbar} E t} \Phi_E(\vec{x}).$$

- Check if there are localized states ( $\Phi_E(\vec{x})$  vanish in spatial infinity). For which  $C$ ? Is energy negative for these states?
- Find the energy spectrum of localized states. How are these energies related to  $mc^2$ ?

- c) Check if there are restrictions on the admissible values of  $C$ ?
- d) Check the transformation properties of solutions with respect to  $\mathbf{C}$ ,  $\mathbf{P}$  and  $\mathbf{T}$  transformations.
- e) Find the non-relativistic limit of solutions (expand in  $c$  around  $c = \infty$ ). Give the first 3 terms of the expansion.

**Hint:** Use the analogy with the nonrelativistic Schrödinger equation for the hydrogen atom. **Do not** derive the form of the spherical functions  $Y_{lm}(\theta, \phi)$ , only use their properties.

**Ex. 3.** We introduce electromagnetic interactions with an external field  $A_\mu(\vec{x}, t)$  defining a *covariant derivative*

$$D_\mu = \partial_\mu + \frac{ie}{\hbar c} A_\mu,$$

where  $e$  is a *charge* of the field. The modified K-G equation has a form

$$(\hbar^2 g^{\mu\nu} D_\mu D_\nu + m^2 c^2) \Psi(\vec{x}; t) = 0.$$

Check that this equation is invariant under a local gauge transformation

$$\begin{aligned} \Psi(\vec{x}, t) &= e^{\frac{ie}{\hbar c} \chi(\vec{x}, t)} \Psi'(\vec{x}, t), \\ A_\mu(\vec{x}, t) &= A'_\mu(\vec{x}, t) - \partial_\mu \chi(\vec{x}, t). \end{aligned}$$

**Ex. 4.** Consider a special case of equation from Ex. 2., for which  $A_\mu(x)$  in some gauge have a form:

$$A_0 = -\frac{Ze}{r}, \quad A_i = 0.$$

Similarly as in Ex.1. find the stationary solutions

$$\Psi_E(\vec{x}; t) = e^{-\frac{i}{\hbar} Et} \Phi_E(\vec{x}).$$

Find the energy spectrum of these localized solutions.

- a) Do we have localized states with positive and negative energy? How does it depend on  $Z$ ?
- b) Are there constraints on  $Z$ ?
- c) Check the transformation of solutions under  $\mathbf{C}$ ,  $\mathbf{P}$  and  $\mathbf{T}$ .
- d) Find the non-relativistic limit of solutions (expand in  $c$  around  $c = \infty$ ). Give the first 3 terms of the expansion.