Quantum Mechanics III, set 10.

Ex. 1. Find the following anticommutators of Dirac field operators in the Schrödinger representation:

$$\{\Psi_\alpha^{(S)}(\vec{x}), \Psi_\beta^{(S)}(\vec{y})\}, \quad \{\Psi_\alpha^{(S)}(\vec{x}), \bar{\Psi}_\beta^{(S)}(\vec{y})\}.$$

Similar excersise in the Heisenberg representation:

$$\{\Psi_\alpha^{(H)}(x), \Psi_\beta^{(H)}(y)\}, \quad \{\Psi_\alpha^{(H)}(x), \bar{\Psi}_\beta^{(H)}(y)\}.$$

Ex. 2. Find a form of the propagator of the Dirac fermion field as a four-dimensional Fourier transform

$$iS_F(x,y) = \langle 0|T\left(\Psi_\alpha^{(H)}(x)\bar{\Psi}_\beta^{(H)}(y)\right)|0\rangle.$$

Show that

$$iS_F(x,y) = T\left(\Psi_\alpha^{(H)}(x)\bar{\Psi}_\beta^{(H)}(y)\right) - :\Psi_\alpha^{(H)}(x)\bar{\Psi}_\beta^{(H)}(y):$$

Ex. 3. Using the generalized Wick theorem calculate

$$\langle 0|T\left(\int d^4x(\bar{\xi}(x)\Psi(x) + \bar{\Psi}(x)\xi(x))\right)^n|0\rangle$$

Consider $\xi(x)$ and $\bar{\xi}(x)$ to be **anticommuting** sources with a property

$$\{\xi_\alpha(x), \xi_\beta(x')\} = \{\bar{\xi}_\alpha(x), \bar{\xi}_\beta(x')\} = \{\bar{\xi}_\alpha(x), \xi_\beta(x')\} = \cdots = 0$$

(here $\alpha,\ \beta$ are spinor indices). Using this result calculate

$$\mathcal{Z}(\xi,\bar{\xi}) = \langle 0|T\exp\left(\int d^4x(\bar{\xi}(x)\Psi(x) + \bar{\Psi}(x)\xi(x))\right)|0\rangle$$

Ex. 4. We define functional derivatives $\delta/\delta\xi_\alpha(x)$ and $\delta/\delta\bar{\xi}_\alpha(x)$ by

$$\frac{\delta}{\delta\xi_\alpha(x)}\xi_\beta(y) = \delta^4(x-y)\delta_{\alpha\beta}-\xi_\beta(y)\frac{\delta}{\delta\xi_\alpha(x)}, \quad \frac{\delta}{\delta\bar{\xi}_\alpha(x)}\bar{\xi}_\beta(y) = \delta^4(x-y)\delta_{\alpha\beta}-\bar{\xi}_\beta(y)\frac{\delta}{\delta\bar{\xi}_\alpha(x)},$$

$$\frac{\delta}{\delta\xi_\alpha(x)}\bar{\xi}_\beta(y) = -\bar{\xi}_\beta(y)\frac{\delta}{\delta\xi_\alpha(x)}, \quad \frac{\delta}{\delta\bar{\xi}_\alpha(x)}\xi_\beta(y) = -\xi_\beta(y)\frac{\delta}{\delta\bar{\xi}_\alpha(x)}$$

Using $\mathcal{Z}(\xi,\bar{\xi})$ from ex. 3. calculate

$$\langle 0|T\left(\Psi_\alpha(x)\bar{\Psi}_\beta(y)\right)|0\rangle, \quad \langle 0|T\left(\Psi_\alpha(x)\bar{\Psi}_\beta(y)\Psi_\gamma(x')\bar{\Psi}_\rho(y)\right)|0\rangle.$$

Hint: Calculate functional derivatives of $\mathcal{Z}$ at $\xi = \bar{\xi} = 0$.

Ex. 5. Consider a system consisting of Dirac field $\Psi, \bar{\Psi}$ and real scalar field ϕ interacting through the interaction Hamiltonian

$$H_{int} = g\int d^3x\bar{\Psi}(x)\phi(x)\Psi(x).$$

Calculate the functional $W(\xi,\bar{\xi},j)$ (connected diagrams) to the order g^2.
Hint: Use the generating functional

$$\mathcal{Z}_0(\xi,\bar{\xi},j) = \exp(W_0(\xi,\bar{\xi},j)),$$

$$W_0(\xi,\bar{\xi},j) = \frac{1}{2}\int dx\int dy j(x)i\Delta_F(x,y;m)j(y)+\int dx\int dy\bar{\xi}(x)iS_F(x,y;M)\xi(y)$$

where m, M are masses of the scalar and Dirac fields, respectively.