Quantum Mechanics III. Set 1.

Ex. 1. Consider an arbitrary non-singular linear homogeneous coordinate transformation

$$x'^{\mu} = A^{\mu}_{\cdot\,\nu}\,x^{\nu}$$

with constant coefficients $A^{\mu}_{\cdot\,\nu}$. Using the property

$$A^{\mu}_{\cdot\,\nu}A_{\rho}^{\cdot\,\nu} = \delta^{\mu}_{\rho}, \quad A_{\mu}^{\cdot\,\nu}A^{\rho}_{\cdot\,\nu} = \delta^{\rho}_{\mu}$$

show that:

a. $c = a_{\mu}b^{\mu}$ - transforms as a scalar.

b. $b_{\nu} = a^{\mu}t_{\mu\nu}$ - transforms as a covariant vector ($t_{\mu\nu}$ - covariant tensor with two covariant indices).

c. $p_{\mu} = \partial/\partial x^{\mu}$ - transforms as a covariant vector.

d. $a_{\mu}b_{\nu}g^{\mu\nu}$ - transforms as a scalar ($g^{\mu\nu}$ - contravariant tensor).

Ex. 2. The Lorentz group $L^{\mu}_{\cdot\,\nu}$, $\mu,\nu = 0,1,2,3$ preserves a form of the "metric tensor" $g^{\mu\nu}$. In the matrix notation $g^{\mu\nu} \to G = \mathrm{diag}(1,-1,-1,-1)$,. Expressing L as a square matrix show that

$$G = L \cdot G \cdot L^{T}.$$

A covariant metric tensor $g_{\mu\nu}$ satisfies $g_{\mu\nu}g^{\nu\rho} = \delta^{\rho}_{\mu}$. Show that

$$L_{\mu}^{\cdot\,\nu} = g_{\mu\alpha}g^{\nu\beta}L^{\alpha}_{\cdot\,\beta}$$

(which means that we can formally "lower" or "rise" indices in L.)

$$L_{\mu\nu}L^{\rho\nu} = \delta^{\rho}_{\mu},$$

where we have used the possibility to lower or rise indices (notice the order!)

Ex. 3. Infinitesimal Lorentz transformations have a form

$$L(\omega)^{\mu}_{\cdot\,\nu} = \delta^{\mu}_{\nu} + \omega^{\mu}_{\cdot\,\nu}.$$

(infinitesimal means that we can neglect terms of the second order in ω when multiplying such matrices). We may also write

$$L(\omega)_{\mu\nu} = g_{\mu\nu} + \omega_{\mu\nu}.$$

Show that the matrix $\omega_{\mu\nu}$ is antisymmetric and in the consequence a full (infinitesimal) matrix is defined by a set of 6 real parameters.
Consider all possible cases when only one parameter is non-zero and the remaining 5 are zero. Find 6 *generators* of the infinitesimal transformations $M^{(\alpha\beta)}$ defined by

$$L^{\mu}_{\cdot\ \nu}{}^{\cdot} = \delta^{\mu}_{\nu} - \frac{i}{2}\omega_{\alpha\beta}M^{(\alpha\beta)\mu}{}_{\cdot\ \nu}{}^{\cdot}.$$

Why there are only 6 of them? Find the *algebra* of these operators (all commutators - they can be expressed as combinations of these generators).
Help: Use operators

$$J_i = \frac{1}{2}\epsilon_{ijk}M^{jk}, \quad K^i = M^{0i}, \quad i,j,k = 1,2,3$$

Ex. 4. Construct transformations corresponding to a finite value of Ω for each case in ex. 3. The construction is based on a multiplication of N identical transformations by $\omega = \Omega/N$ and using the identity:

$$\lim_{N\to\infty}(\mathbf{1} + \frac{\mathbf{x}}{N})^N = \exp(\mathbf{x}),$$

(exponential of a matrix means a sum of the power series of the exponent:

$$\exp(\mathbf{A}) = \mathbf{1} + \sum_{n=1}^{\infty}\frac{\mathbf{A}^n}{n!}.$$