# Quantum Mechanics III, set 2.

**Ex. 1.** Consider special solutions of Klein-Gordon equation

$$(\hbar^2 g^{\mu\nu}\partial_\mu\partial_\nu + m^2c^2)\Psi(\vec{x};t) = 0,$$

in a form

$$\Psi^{\pm}_{\vec{p}}(\vec{x};t) = e^{\mp\frac{i}{\hbar}E_p t + \frac{i}{\hbar}\vec{p}\vec{x}}, \quad E_p = +\sqrt{\vec{p}^2c^2 + m^2c^4}.$$

Check the norm of these states, i.e. calculate

$$\int d^3x i\hbar \left(\Psi^{\pm}_{\vec{q}}{}^*(\vec{x};t)(\partial_t\Psi^{\pm}_{\vec{p}}(\vec{x};t)) - (\partial_t\Psi^{\pm}_{\vec{q}}{}^*(\vec{x};t))\Psi^{\pm}_{\vec{p}}(\vec{x};t)\right).$$

a) How this norm transforms under orthochronous Lorentz transformations?

b) What are the transformation rules for: $d^3p$, $d^3p/(2E_p)$?

c) Consider a general solution of the Klein-Gordon equation in a form

$$\Phi(\vec{x},t) = \int (dp)\left(a(\vec{p})\Psi^{+}_{\vec{p}}(\vec{x},t) + b^*(\vec{p})\Psi^{-}_{\vec{p}}(\vec{x},t)\right)$$

where

$$(dp) = \frac{d^3p}{(2\pi\hbar)^3(2E_p)}$$

Find a norm of this state.

**Ex. 2.** We introduce interaction to the K-G equation:

$$(\hbar^2 g^{\mu\nu}\partial_\mu\partial_\nu + m^2c^2 + W(\vec{x},t))\Psi(\vec{x};t) = 0,$$

where $W(\vec{x},t)$ is a given function. Let us assume a particular form of this function

$$W(\vec{x};t) = -\frac{C}{r}$$

(which means it is time independent and spherically symmetric, with some constant $C$). We are looking for special solutions of this equation in a form

$$\Psi_E(\vec{x};t) = e^{-\frac{i}{\hbar}Et}\Phi_E(\vec{x}).$$

a) Check if there are localized states ($\Phi_E(\vec{x})$ vanish in spatial infinity). For which $C$? Is energy negative for these states?

b) Find the energy spectrum of localized states. How are these energies related to $mc^2$?

c) Check if there are restrictions on the admissible values of $C$?

d) Check the transformation properties of solutions with respect to **C**, **P** and **T** transformations.

e) Find the non-relativistic limit of solutions (expand in $c$ around $c = \infty$). Give the first 3 terms of the expansion.

**Hint:** Use the analogy with the nonrelativistic Schrödinger equation for the hydrogen atom. **Do not** derive the form of the spherical functions $Y_{lm}(\theta, \phi)$, only use their properties.

**Ex. 3.** We introduce electromagnetic interactions with an external field $A_\mu(\vec{x}, t)$ defining a *covariant derivative*

$$D_\mu = \partial_\mu + \frac{ie}{\hbar c} A_\mu,$$

where $e$ is a *charge* of the field. The modified K-G equation has a form

$$(\hbar^2 g^{\mu\nu} D_\mu D_\nu + m^2 c^2)\Psi(\vec{x}; t) = 0.$$

Check that this equation is invariant under a local gauge transformation

$$\Psi(\vec{x}, t) = e^{\frac{ie}{\hbar c}\chi(\vec{x},t)}\Psi'(\vec{x}, t),$$

$$A_\mu(\vec{x}, t) = A'_\mu(\vec{x}, t) - \partial_\mu \chi(\vec{x}, t).$$

**Ex. 4.** Consider a special case of equation from Ex. 2., for which $A_\mu(x)$ in some gauge have a form:

$$A_0 = -\frac{Ze}{r}, \quad A_i = 0.$$

Similarly as in Ex.1. find the stationary solutions

$$\Psi_E(\vec{x};t) = e^{-\frac{i}{\hbar}Et}\Phi_E(\vec{x}).$$

Find the energy spectrum of these localized solutions.

a) Do we have localized states with positive and negative energy? How does it depend on $Z$?

b) Are there constraints on $Z$?

c) Check the transformation of solutions under **C**, **P** and **T**.

d) Find the non-relativistic limit of solutions (expand in $c$ around $c = \infty$). Give the first 3 terms of the expansion.