Real and complex analysis

Project

,

Andrzej Jarynowski, Agata Wojciechowska, Jarosław Mrugała and Krzysztof Kaczan

European Masters Education
in Industrial Mathematics

12 February 2008

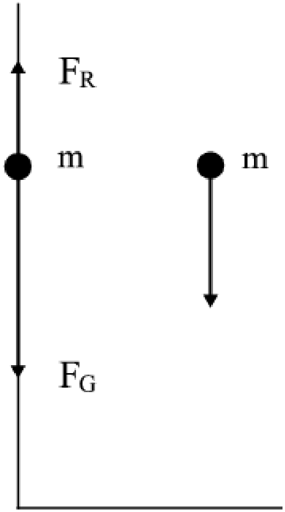
Table of contents

The problem

- The real-life problem
- The general question:
  If someone fall a distance of about 30 ft, what will the impact speed be?

Physical values

- distance fallen x
- time t
- velocity v
- acceleration a
- mass m



Relations between the variables

Velocity

$$v = \frac{dx}{dt}$$

Acceleration

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$$

or

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}.$$

The forces

The forces acting on the body:

- gravitation force $F_G = mg$
- air resistance force F_R

Constructing the model

From the Newton's Second Law, the total force acting on the body

$$F = F_G + F_R$$

is equal to the acceleration

$$ma = m\frac{dv}{dt},$$

that yields

$$mg - F_R = m\frac{dv}{dt}$$

or

$$mg - F_R = mv\frac{dv}{dx}$$

F_R

The air resistance force F_R is not distance or time dependent but it is directly proportional to the velocity:

$$F_R = kv$$

It should be taken into account what kind of physical body is falling.
Let's take the general relation $F_R = kv^n$

F_R

$$mg - kv^n = mv\frac{dv}{dx}$$

$$g - Kv^n = v\frac{dv}{dx},$$

where $K = \frac{k}{m}$.

What n should we choose?

1. the air resistance is directly proportional to the speed for small compact objects n=1
2. the air resistance is directly proportional to the square of the speed if we are examining objects of the size of human body n=2

Therefore $n = 2$, so

$$g - Kv^2 = v\frac{dv}{dx}$$

Terminal speed

The terminal speed of a human body is about $v = 53,62\frac{m}{s}$.
In this situation there is no acceleration, that means

$$g - kv^2 = 0.$$

Substituting $g = 9,8065$ and $v = 53,62$ we get

$$K = 0,00341$$

The final equation

The following equation describes the motion of the body falling out of a window:

$$g - 0,00341 v^2 = v \frac{dv}{dx}$$

$$x = \frac{1}{2K} \ln\left(\frac{g}{g - Kv^2}\right). \tag{1}$$

Since $m\frac{dv}{dt} = mg - kv^2$ then after longer integration we get

$$v = \frac{1}{k}\sqrt{gh}\tanh t\sqrt{gh}$$

[1] Substituting v in (1) we get

$$x(t) = \frac{1}{K}\ln[\cosh(t\sqrt{gh})]$$

If $x = 30\text{ft}$ then the impact speed is $29,52\frac{miles}{h}(\approx 47,5\frac{km}{h})$

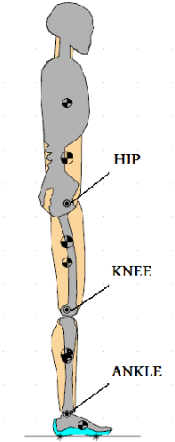
[1]

$$\sinh = \frac{e^x - e^{-x}}{2} \qquad \cosh = \frac{e^x + e^{-x}}{2} \qquad \tanh = \frac{\sinh}{\cosh}$$

Another point of view

Let's consider a human body as a solid with three oscillation points placed in:

- hip
- knee
- ankle



The energy of the system

The energy of the impact into the three oscillators

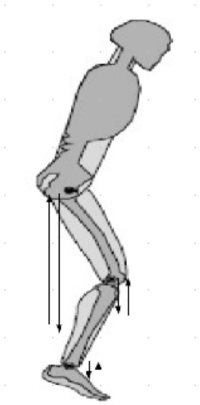
$$E = E_h + E_k + E_a,$$

where

$$E_h \geq E_k \geq E_a$$

The energy of the hit is equal to the potential energy which is described by the formula:

$$E = mgh$$



The energy of the system

The oscillation energy

$$E = \frac{kx^2}{2},$$

where k - stiffness coefficient.
$x = b\sqrt{E}$, where $b = \sqrt{\frac{2}{k}}$
When $x > x_{max}$ then the oscillator is destroyed.

The ground

The ground on which the man falls can absorb some energy.

Coefficient of restitution properties of impact surfaces.

impact surface	COR
playground foam	0.57
carpet	0.33
linoleum	0.12
wood	0.12

The 'efficient' energy $E = E(1 - COR)$

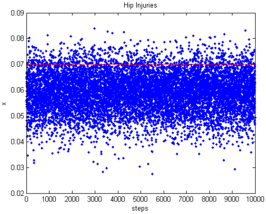
Statistics

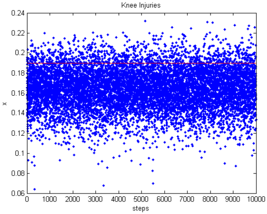
- statistical data: 1643 fall cases
- proportion of injuries:
  - hip $5,05\%$ pelvis
  - knee $9,07\%$ tibia, fibula
  - ankle $6,7\%$ metatarsal
- the falls from a height above 4.5 feet

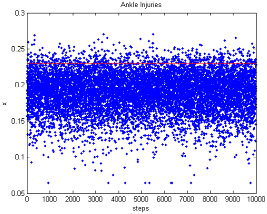
Using the empirical data we try to optimise k in such way that the proportion of the cases of the joint injured is proportional to the empirical data. How to divide the energy?

```
EH=E*max(normrnd(0.45,0.12),0.05);
EK=(E-EH/2)*(1+normrnd(0.1,0.1));
EA=E-EH-EK
```

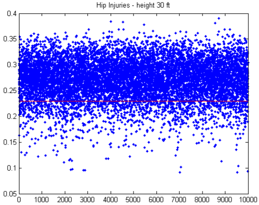
Estimation of k

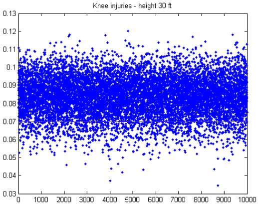


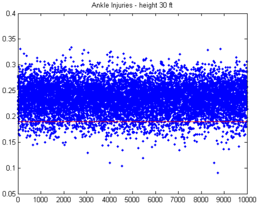




Probability of getting injured if the height is 30 ft



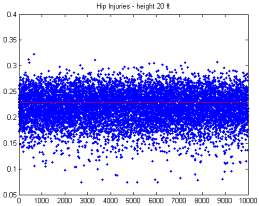


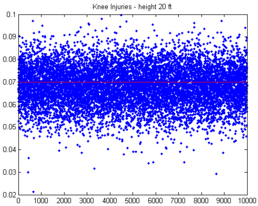


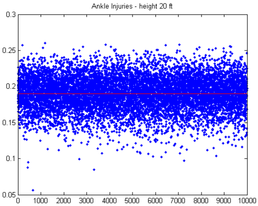
$0,8677$ $0,9347$ $0,9067$

Probability of getting injured: 0,992

Probability of getting injured if the height is 20 ft



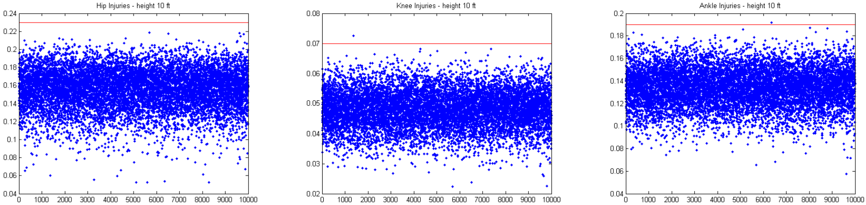




$0,3888$ $0,5121$ $0,4211$

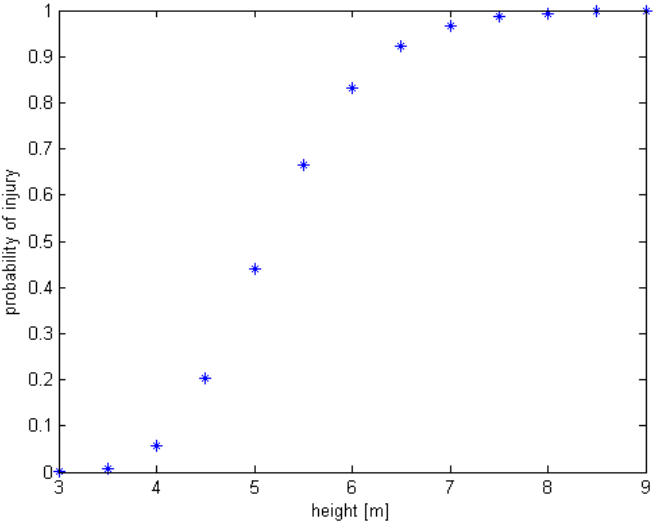
Probability of getting injured: 0,8274

Probability of getting injured if the height is 10 ft



Probability of getting injured: ≈ 0

Probability of injury dependent of high



Conclusion

Aspects to work out:

- the trajectory of the fall
- the mass of the man
- different grounds

If you fall about 30ft, you hit the ground at about $30\frac{miles}{h}$. This could be compared to a car crash at at least $30\frac{miles}{h}$. In most cases such crashes causes injuries. The second method shows also that it is almost impossible that the impact will cause no injury.

Perhaps a soft landing in a flower bed might allow the criminal to get away unhurt...