

The Thermodynamics of White Dwarf Matter

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Received May 20, 1970

An equation of state for white dwarf matter of high density and finite temperature, in which the ions form a cubic body centered lattice, is derived; some thermodynamic quantities are discussed with numerical examples. The passage from gas to solid is treated as a quasi-static phase transition and the effect of latent heat release on stellar evolution is briefly discussed.

Key words: white dwarf — white dwarf matter — crystallization

1. Introduction

It has been pointed out by Kirzhniz (1960), Abrikosov (1961) and Salpeter (1961) that the classical, "black dwarf" equation of state for cold matter of high density should be modified to take account of Coulomb interactions: high density reduces the electrostatic shielding of ions by electrons, and as a result the ions arrange themselves in a minimum-energy configuration — a body-centered cubic lattice (Carr, 1961). Salpeter (1961) has derived an equation of state that takes account of this phenomenon in zero-temperature matter. Black dwarfs based on this equation of state were calculated by Hamada and Salpeter (1961).

The evolution of *white* dwarfs requires the analogous treatment of finite temperature matter, and this question has recently received qualitative consideration by Mestel and Ruderman (1967) and by van Horn (1968). A detailed treatment of the thermodynamics of solid white dwarf matter will be given in Section 2 of this paper.

The question of latent heat release upon solidification has been the subject of some controversy (Mestel and Ruderman, 1967; van Horn, 1968.) In Section 3 we propose a simple recipe for dealing with the transition of ions from the gaseous to the solid phase: the transition is regarded as a quasi-static phase transition and is calculated by equating

the chemical potentials in the two phases. The resulting transition curve in the (T, ρ) diagram is, perhaps surprisingly, rather similar to the one obtained by Mestel and Ruderman (1967) from a quite different hypothesis (Lindemann's rule). The latent heat is appreciable and is mainly due to the change in internal energy, since the volume change is only of the order of 1%.

The rest of the paper is devoted to some specific examples and a few conclusions regarding the heat capacities of white dwarfs (Section 4).

2. Thermodynamic Quantities for the Solid Phase

In deriving the thermodynamic quantities for the solid phase we may regard the zero-temperature terms (Salpeter, 1961) as "zero-point" terms, to be added subsequently to the temperature dependent expression connected with the oscillations of the ion lattice.

All temperature dependent quantities can be derived by differentiation from the free energy

$$F = N k T \sum_{\lambda} \langle \frac{1}{2} \beta \hbar \omega_{k\lambda} + \log (1 - e^{-\beta \hbar \omega_{k\lambda}}) \rangle, \quad (1)$$

where N and T are the total number of ions and the temperature; k and \hbar are Boltzmann's and Planck's constants; $\beta = (k T)^{-1}$ and $\omega_{k\lambda}$ is the circular frequency, a function of the wave vector k and the excitation mode λ . The average in (1) is over the first

Brillouin zone (say). With one exception, $\omega_{k\lambda}$ is proportional to the ion plasma frequency.

$$\Omega_p = (4 \pi Z^2 e^2 n_z / M_z)^{1/2}, \quad (2)$$

where Ze , n_z and M_z are the ionic charge, number density and mass. The exception is the longitudinal (plasmon) branch at long wavelengths, $k^{-1} > k_{sc}^{-1} = r_{sc}$, the screening radius. In this case we have

$$\omega^2 = \Omega_p^2 / (1 + k_{sc}^2 / k^2) \quad (3)$$

(Pines, 1963, p. 243).

The magnitude of k_{sc} depends on the density of the degenerate electrons. In the relativistic limit, when the electronic Fermi energy $E_F \geq mc^2$, a simple Thomas-Fermi calculation yields

$$k_{sc}^2 = 12 \pi n e^2 / E_F, \quad (4)$$

where n is the electronic number density. (The ion-relativistic expression has a coefficient of 6 instead of 12.) The relativistic case is the relevant one for the solid phase (since we must have $r_{sc} \geq r_i$, where r_i is the interionic distance). Since $E_F \propto n^{1/3}$ we have according to (4), $k_{sc} \propto n^{1/3}$, so that

$$n k'_{sc}(n) = \frac{1}{3} k_{sc}. \quad (5)$$

The differentiation of (3) results in

$$n \omega'(n) = \frac{1}{2} g(k) \omega, \quad g(k) = 1 - \frac{2}{3} (1 + k^2/k_{sc}^2)^{-1}. \quad (6)$$

In the absence of screening ω is proportional to Ω_p and (6) holds with $g(k) = 1$.

Furthermore,

$$n \frac{\partial}{\partial n} g(k) = h(k) = -\frac{4}{9} (1 + k_{sc}^2/k^2)^{-1}. \quad (7)$$

We introduce the notation

$$\varpi = \omega / \Omega_p, \quad k \theta = \hbar \Omega_p. \quad (8)$$

Using the foregoing formulae we obtain from (1) the following expressions for the pressure P , the entropy per ion s and the energy per ion u :

$$P = \frac{1}{2} n_z k T \sum_{\lambda} \left\langle g(k) \left(\frac{1}{2} \varpi \theta / T + \frac{\varpi \theta / T}{e^{\varpi \theta / T} - 1} \right) \right\rangle, \quad (9)$$

$$s/k = \sum_{\lambda} \left\langle \frac{\varpi \theta / T}{e^{\varpi \theta / T} - 1} - \log(1 - e^{-\varpi \theta / T}) \right\rangle, \quad (10)$$

$$u = k T \sum_{\lambda} \left\langle \frac{1}{2} \varpi \theta / T + \frac{\varpi \theta / T}{e^{\varpi \theta / T} - 1} \right\rangle. \quad (11)$$

The averages in (9) – (11) can be calculated by the method of Cohen and Keffer (1955), which is

based on the values of $\varpi_{k\lambda}$ at 29 points in the first Brillouin zone, and (because of the lattice symmetries) is equivalent to numerical integration using 512 points in that zone. The values of $\varpi_{k\lambda}$ at these 29 points were given by Carr (1961, Table 1; the $\varpi_{k\lambda}$ are obtained by dividing the entries in Carr's table by $\sqrt{3}$) and should be corrected, when necessary, for screening. In (9), $g(k)$ is given by (6) for the longitudinal branch is $k \geq k_{sc}$; otherwise $g(k) = 1$.

The chemical potential is

$$\mu_z = u - sT - P/n_z. \quad (12)$$

The following thermodynamic quantities are also needed in evolutionary calculations:

$$\frac{\partial P(n_z, T)}{\partial T} = \frac{1}{2} n_z \sum_{\lambda} \left\langle g(k) \left(\frac{\varpi \theta / T}{e^{\varpi \theta / T} - 1} \right)^2 e^{\varpi \theta / T} \right\rangle, \quad (13)$$

$$\begin{aligned} \frac{\partial P(n_z, T)}{\partial n_z} &= P/n_z + \frac{1}{2} k T \sum_{\lambda} \left\langle -h(k) \left(\frac{1}{2} \varpi \theta / T + \frac{\varpi \theta / T}{e^{\varpi \theta / T} - 1} \right) \right. \\ &\quad \left. + \frac{1}{2} g^2(k) \left[\frac{1}{2} \varpi \theta / T + \frac{\varpi \theta / T}{e^{\varpi \theta / T} - 1} - \left(\frac{\varpi \theta / T}{e^{\varpi \theta / T} - 1} \right)^2 e^{\varpi \theta / T} \right] \right\rangle, \quad (14) \end{aligned}$$

$$\frac{1}{k} \frac{\partial s(n_z, T)}{\partial T} = \frac{1}{T} \sum_{\lambda} \left\langle \left(\frac{\varpi \theta / T}{e^{\varpi \theta / T} - 1} \right)^2 e^{\varpi \theta / T} \right\rangle, \quad (15)$$

$$\begin{aligned} \frac{1}{k} \frac{\partial s(n_z, T)}{\partial n_z} &= -\frac{1}{2 n_z} \sum_{\lambda} \left\langle g(k) \left(\frac{\varpi \theta / T}{e^{\varpi \theta / T} - 1} \right)^2 e^{\varpi \theta / T} \right\rangle, \quad (16) \end{aligned}$$

where $g(k)$ and $h(k)$ (Eq. (7)) are used in the same sense as $g(k)$ in (9): in the absence of screening $h(k) = 0$.

The above formulae must be supplemented by the zero-temperature terms (Salpeter, 1961) and by the electronic and radiative contributions.

3. The Gas-solid Transition

The thermodynamic quantities of the foregoing Section describe an extreme phase of white dwarf matter. The other, low density, extreme case is obtained by treating the ions as a perfect gas. Somewhere in between we expect a "liquid" phase. Very little is known about this complicated phase (except for its low density region, where the Debye-Hückel theory should provide a good approximation), and it does not seem possible at present to write down

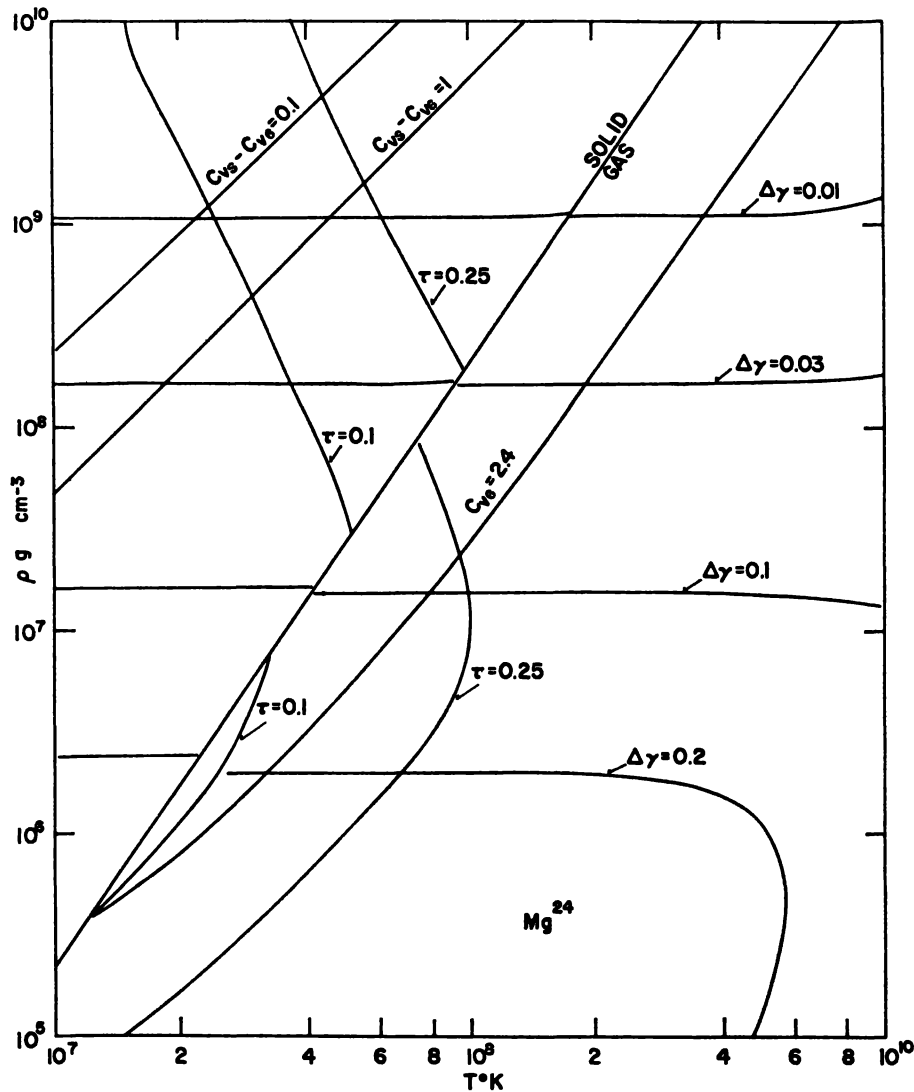


Fig. 1. The equation of state for Mg^{24} . Contours of $\Delta\gamma = \gamma - 4/3$, the excess of the adiabatic index over $4/3$; the specific heat per particle at constant volume C_{vg} in the gaseous state; the excess $C_{vs} - C_{vg}$ of the solid over the gaseous specific heat in the solid phase; and the thermal runaway parameter τ . The transition line between the two phases [is also given]

expressions for its thermodynamic properties with any reasonable degree of confidence. The simplest way out of this difficulty is to ignore the liquid phase altogether and to replace it by a discontinuous transition from gas to solid. Mestel and Ruderman (1967) used a hypothesis similar to Lindemann's rule (Pines, 1963) to define for every density ρ a transition temperature $T_g \cong 10^7 Z^{5/3} (\rho/10^6)^{1/3}$ K. But classical thermodynamics lead to a unique condition for the transition between two adjacent phases in a stellar model (in which pressure and temperature are continuous): the chemical potential must be continuous

across the boundary between the two phases. And if both phases are electrically neutral, the condition is expressed by the constancy of $\mu_z + z\mu_e$, where μ_z and μ_e are the ionic and electronic chemical potentials, respectively. Since these quantities are known (Eq. (12)), we can calculate the thermodynamic variables at the transition. In particular, we can determine the densities of the two phases and the latent heat. Numerical examples are given in the next Section.

It should be emphasized that this method of treating the transition between the gaseous and solid phases results from the *ad hoc* assumption that the

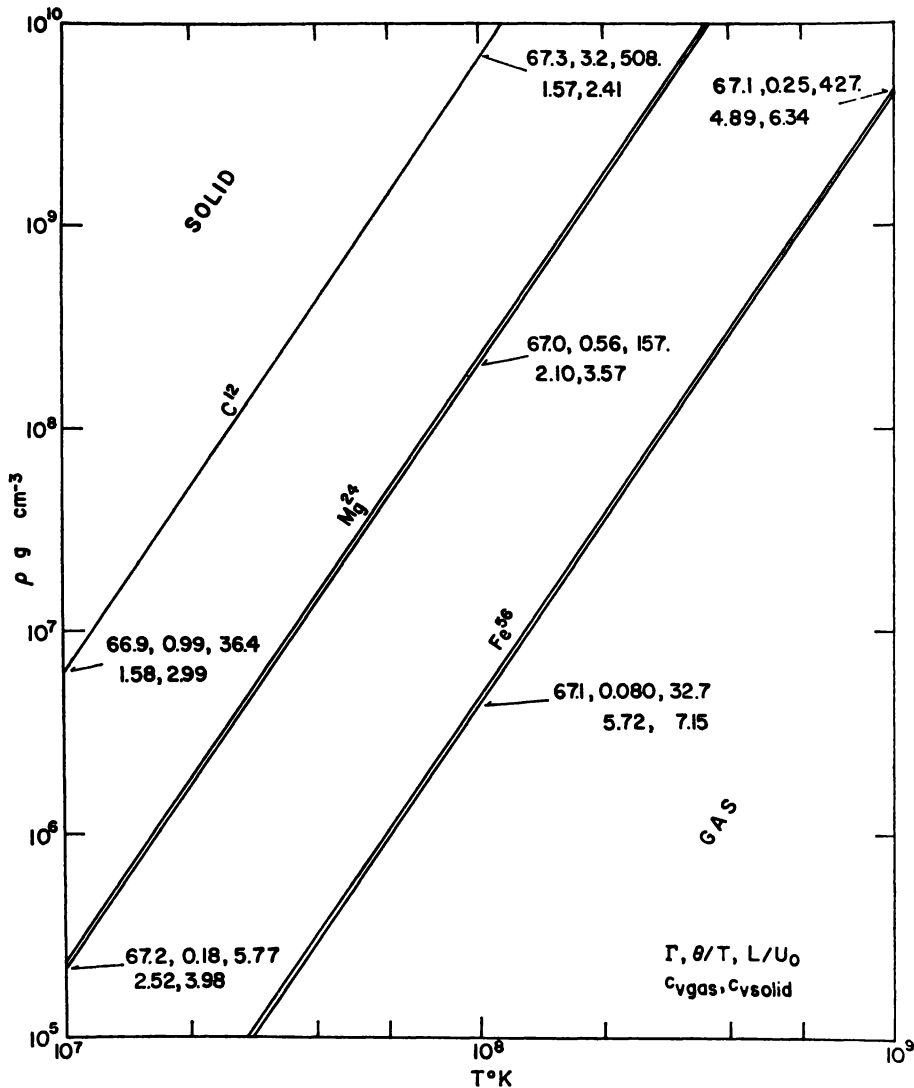


Fig. 2. Transition lines for C^{12} , Mg^{24} and Fe^{56} . The values of the Coulomb parameter Γ , the ratio θ/T of the Debye temperature $\theta = h\Omega/k$ to the temperature, the latent heat per gram in units of $U_{\odot} = \frac{GM_{\odot}}{R_{\odot}} = 1.9 \times 10^{15} \text{ erg g}^{-1}$, and the specific heats per ion are shown at selected points along the transition lines. (Note that U_{\odot} divided by the solar luminosity defines the solar Kelvin-Helmholtz time)

liquid phase can be ignored. (Inclusion of the Debye-Hückel effects in the regime $0 < \Gamma = Ze/(kT r_i) \ll 1$ would hardly be an improvement. The gap between $\Gamma \cong 1$, which characterizes a dense Debye-Hückel gas, and $\Gamma \cong 50$, typical of the solid phase, still remains). The only justification for such an assumption lies in its simplicity and the fact that it enables us to construct evolutionary models for white dwarfs. Any possible merits it may have, must be judged on the basis of a comparison between such models and observation.

4. Numerical Examples

Figure 1 shows the results of numerical evaluation of the equation of state for Mg^{24} . Contours of $\gamma - 4/3$, specific heats $C_v = T(\partial s/\partial T)_v$ and the thermal runaway parameter $\tau = \frac{\rho}{PT} \left(\frac{\partial P}{\partial s} \right)_e \left(\frac{\partial T}{\partial \rho} \right)_s (\gamma - 4/3)^{-1}$ (Rakavy and Shaviv, 1967) are shown on both sides of the transition line. All these quantities are discontinuous: there is a decrease in $\gamma - 4/3$ on passing from gas to solid, due to the Coulomb interaction, which "has a γ of $4/3$ ". The difference

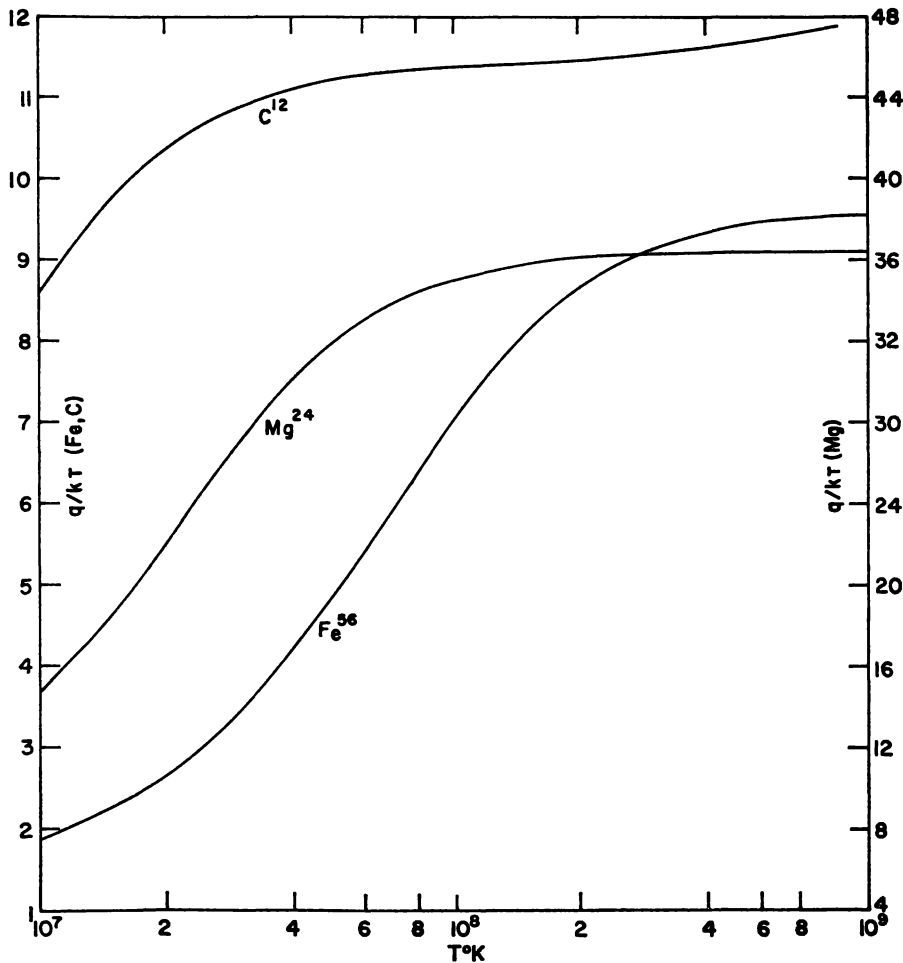


Fig. 3. The ratio of the latent heat per ion to kT for C^{12} , Mg^{24} and Fe^{56} . The values for C^{12} should be multiplied by 10

$C_{vs} - C_{vg}$ between the specific heats per ion calculated for the ionic solid (C_{vs}) and the ionic perfect gas (C_{vg}) increases with temperature at constant density; it approaches the Dulong and Petit value $3/2$ at low densities; at higher densities the Debye temperature θ at the transition is not small with respect to T . As for the runaway parameter τ , it decreases slightly upon solidification, implying that the solid state is more susceptible to thermal runaway.

Figure 2 shows the transition lines for C^{12} , Mg^{24} and Fe^{56} , with numerical values of some thermodynamical quantities at selected points. In a (T, ρ) -diagram the transition is defined by two lines, between which both phases exist in a definite ratio determined by T and v (for a given quantity of mass). In the case of C^{12} the lines practically coincide; in the other cases the fractional change $(\rho_{solid} - \rho_{gas})/\rho_{gas}$ is 1–3%, which

is much larger than van Horn's (1968) estimate ($\delta\rho/\rho \sim 10^{-4}$). It will be seen that $\Gamma = Ze/(kTr_i)$ is fairly constant, of order 70, along the transition lines, vindicating Mestel and Ruderman's (1967) formula for the melting temperature. The difference between $C_{v, solid}$ and $C_{v, gas}$ is close to 1.5, but the specific heats per ion themselves are *not* 1.5 and 3, respectively. This is due to the electronic and radiative contribution. In the case of Mg^{24} at $T = 10^7$ K and $\rho = 2 \times 10^5$ g/cc, for example, 40% of the $C_{v, gas} = 2.52$ are due to the 12 electrons per ion; the corresponding contribution to $C_{v, solid} = 3.98$ is 26%. The radiative contributions are negligible; they may be important in other cases because they vary as AT^3/ρ . In the case of Fe^{56} at $T = 10^8$ K and $\rho = 4 \times 10^6$ g/cc the 26 electrons contribute 74% to $C_{v, gas} = 5.72$ and 59% to $C_{v, solid} = 7.15$.

The latent heat per gram L (given in units of $u_{\odot} = GM_{\odot}/R_{\odot}$) varies rather quickly along the transition lines. It is obtained by evaluating the expression $\Delta u + P\Delta\rho^{-1}$ at the transition and is therefore dependent on the manner in which the latter takes place, in particular on the assumption of a discontinuous phase change (Section 3). However, this does not seem to be crucial, because the density change at the transition is small ($\sim 1\%$) and so L is practically equal to its minimal value Δu , which consists almost entirely of the electrostatic lattice energy. Now, whatever the actual form of transition from perfect ion gas to ionic lattice may be, energy of this amount must somehow be extracted, continuously or suddenly: it cannot be ignored. The members given in Figure 2 suggest that it would take a star of $0.5 M_{\odot}$ with a luminosity of 10^3 – $10^4 L_{\odot}$ (typical of the nuclei of planetary nebulae) 10^3 – 10^5 years to get rid of the extra energy before it can settle down to a solid state. It should, however, be noted that since L changes rapidly along the transition lines (Fig. 3) it will vary from star to star, and also within a given star: the evolutionary tracks

of different stars will cross the transition lines at different points (depending on the total mass), and the same is, of course, true of different mass shells in a given star.

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