Stochstic processes and applications. Problem set 1.

Ewa Gudowska-Nowak* and Paweł F. Góra†

M. Smoluchowski Institute of Physics,

Jagiellonian University, 30-059 Kraków, Reymonta 4, Poland.

A Markov chain model.

Consider two pots A and B with three red balls and two white balls distributed between them so that A always has two balls and B always has three balls. There are three different configurations for the pots as shown in figure below. We obtain transitions between these three configurations by picking a ball out of A at random and one out of B at random and interchanging them. Compute the transition matrix $\mathbf{Q}$ of elements $Q_{m,n} \equiv P(n, s+1|m, s)$. Here $P(n, s|m, s_0)$ stands for the conditional probability that a system will jump (in one step) to state m at time s provided it occupied state m at time s_0. The Chapman-Kolmogorov equation for this problem reads

$$P(n, s+1|n_0, s_0) = \sum_{m=1}^{M} P(n, s+1|m, s)P(m, s|n_0, s_0) \tag{1}$$

and its formal solution is obtained by iteration $P(n, s|m, s_0) = (\mathbf{Q})^{s-s_0}_{m,n}$ where the RHS denotes the (m,n)th element of the matrix $\mathbf{Q}$ raised to the $s - s_0$ power. The probability $P(n, s)$ is thus given by

$$P(n, s) = \sum_{m=1}^{M} P(m, 0)(\mathbf{Q})^{s}_{m,n} \tag{2}$$

In general, the transition matrix is not a symmetric matrix, therefore its right and left vectors are different (although complete and orthogonal). If the transition matrix is regular (i.e. if all elements of some power $(\mathbf{Q})^N$ are nonzero), the probability $P(n, s)$ tends to a unique stationary state after a long time. If initially $P(1, s_0 = 0) = 1$, $P(2, s_0 = 0) = 0$, $P(3, s_0 = 0) = 0$, compute the probability $P(n, s)$ ($n = 1, 2, 3$ at time s). Calculate

$$< y(s) >= \sum_{n=1}^{3} nP(n, s) \tag{3}$$

Find $\lim_{s\to\infty} P(n, s) = P_{st}$.

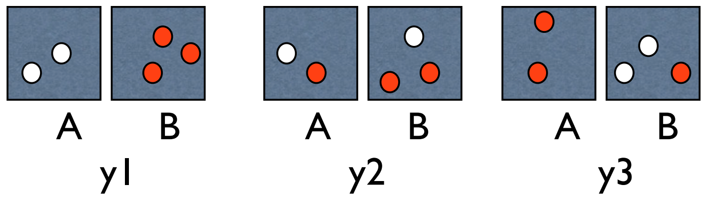

FIG. 1. States y_1, y_2, y_3 of the system (equivalent to $(1, 2, 3)$).

Neyman-type A distribution.

This distribution belongs to the category of cluster (self-exciting) point processes and has found application in aerospace engineering, ecology, reliability, and forestry. Cluster processes are characterized by a primary (mother) process which generates at each point secondary (daughter) events. When only daughter events appear in the final process and when primary and secondary distributions are both Poisson, the resulting distribution is known as the Neyman type-A counting distribution with probability function

$$P(n, \alpha, \beta) = \sum_{m=0}^{\infty} \frac{(\alpha m)^n e^{-\alpha m}}{n!} \frac{\beta^m e^{-\beta}}{m!} \tag{4}$$

Find the first and the second moment of this distribution. Generate frequency histograms of $P(n, \alpha, \beta)$. Check modality of the distribution for $\beta >> \alpha$ and $\beta << \alpha$.

Birth and death process

Consider a process with a constant rate of producing new events λ and a constant rate of destructing (annihilating) μ. By definition $p_k(t)$ stands for the probability that we observe k events in time interval t. which satisfies equations

$$\begin{aligned} dp_0/dt &= -\lambda p_0(t) + \mu p_1(t) \\ dp_k/dt &= -(\lambda + \mu) p_k(t) + \lambda p_{k-1}(t) + \mu p_{k+1}(t) \end{aligned} \tag{5}$$

Find equilibrium solution for this problem.

*gudowska@th.if.uj.edu.pl
†gora@if.uj.edu.pl

REFERENCES

[1] R.D. Astumian, *Thermodynamics and kinetics of a Brownian motor*, Science **276** 917-22 (1997).

[2] J.-P. Bouchaud and A. Georges *Anomalous diffusionin disordered media: statistical mechanics, models and physical applications*, Physics Reports **195** 127-293 (1990).

[3] W. Feller *An introduction to probability theory and its application*, Wiley (1966).

[4] E. Montroll and M. F. Shlesinger, *On 1/f noise and other distributuions with long tails*, Proc Natl. Acad. Sci. **79** 3380-3383 (1982)

[5] M. Kimura, *Solution of a process of random genetic drift with a continuous model*, Proc. Natl. Acad. Sci. **41** 144-50 (1955).

[6] N.G. van Kampen, *Stochastic processes in physics and chemistry*, North Holland (1981).

[7] M. Marsili and Y-C. Zhang, *Fluctuations around Nash equilibria in game theory*, Physica A **245**181-188 (1997).

[8] W. Ebeling and I.M. Sokolov, *Statistical Thermodynamics and Stochastic Theory of Nonequilibrium Systems*, World Scientific (2005).

[9] E. W. Montroll and M. F. Shlesinger *Maximum entropy formalism, fractals, scaling phenomena, and 1/f noise: A tale of Tails* , J. Stat. Phys. **32** 209-230 (1983).

[10] J.Klafter, S. C. Lim, R. Metzler, Eds.*Fractional dynamics: Recent Advances*, World Scientific (2012).

[11] H. Risken *The Fokker-Planck equation; methods of solutions and applications*, Springer Verlag (1989).

[12] M.B. Marconi, A. Puglisi, L. Rondoni, A. Vulpiani *Fluctuation-dissipation: Response Theory in Statistical Physics*, Phys. Rep. **461** 111-195 (2008).