Convergence to an Equilibrium for Wave Maps on a Curved Manifold

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Introduction

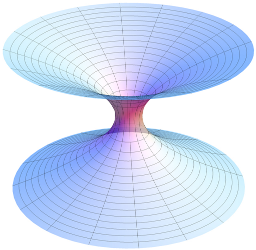
- Part of a long-term project on global dynamics of nonlinear wave equations (power nonlinearities, **wave maps**, Yang-Mills)
- Goal: understanding of relaxation to a stationary equilibrium on an unbounded domain through **the dissipation of energy by dispersion**
- Our motivation comes from general relativity, in particular the problem of formation of black holes in gravitational collapse
- We want to construct a toy-model of which is as simple as possible:
  - scalar semilinear equation with globally regular solutions
  - unique nontrivial stationary equilibrium
  - no unstable, neutral, or oscillatory modes
- In order to design a model with these properties we consider equivariant wave maps with a suitably chosen curved domain

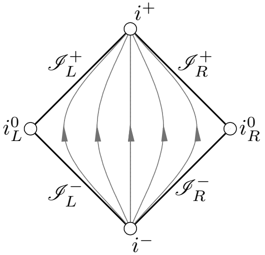
Model

- Wave maps $u : \mathscr{M} \mapsto \mathscr{N}$ are critical points of the action
$$S = \int g^{\alpha\beta} \partial_\alpha X^A \partial_\beta X^B G_{AB} \sqrt{-g}\, dx\,,$$
where $(\mathscr{M}, g_{\alpha\beta})$ is a Lorentzian and $(\mathscr{N}, G_{AB})$ a Riemannian manifold.
- The wave map equation: $\Box_g X^A + \Gamma^A_{BC}(X) \partial_\alpha X^B \partial_\beta X^C g^{\alpha\beta} = 0$
- Domain: $\mathscr{M} = \{(t,r) \in \mathbb{R}^2, (\vartheta, \varphi) \in S^2\}$ with metric

$$g_{\alpha\beta} dx^\alpha dx^\beta = -dt^2 + dr^2 + (r^2 + a^2)\,(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)$$

- The hypersurfaces $t =$ const have two asymptotically flat ends at $r \to \pm\infty$ connected by a neck of area $4\pi a^2$ at $r = 0$ (wormhole spacetime)





Equivariant wave map equation

- Target: $\mathscr{N} = \mathbb{S}^3$ with the round metric

$$G_{AB}dX^A dX^B = dU^2 + \sin^2 U(d\Theta^2 + \sin^2\Theta\, d\Phi^2)$$

- Equivariant ansatz: $U = U(t,r), (\Theta, \Phi) = \Omega_\ell(\theta, \phi)$, where $\Omega_\ell : S^2 \mapsto S^2$ is a harmonic eigenmap map with eigenvalue $\ell(\ell+1)$ ($\ell \in \mathbb{N}$).

- Semilinear scalar wave equation

$$U_{tt} = U_{rr} + \frac{2r}{r^2+a^2}\, U_r - \frac{\ell(\ell+1)}{2}\, \frac{\sin(2U)}{r^2+a^2}$$

- The length scale a plays two roles:
  - breaks scale invariance
  - removes the singularity at $r = 0 \Rightarrow$ global-in-time regularity
- Recall that for $a = 0$ (flat domain $\mathbb{R}^{3+1}$) we have
  - small data global-in-time existence and asymptotic decay to vacuum
  - large data self-similar blowup in finite time

Static solutions

- Conserved energy

$$E = \frac{1}{2}\int_{-\infty}^{\infty} \left((r^2+a^2)(U_t^2+U_r^2)+\ell(\ell+1)\sin^2 U\right) dr$$

- Finiteness of energy requires that $U(t,-\infty)=m\pi$, $U(t,\infty)=n\pi$. We choose $m=0$ so n determines the topological degree of the map
- For each n there exists a unique static solution $U_n(r)$ (harmonic map) which minimizes the energy in its class (proof: shooting argument).

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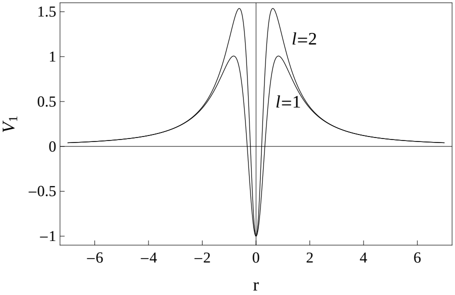
Linear perturbations

- Substituting $U(t,r) = U_n(r) + e^{\lambda t}(r^2+a^2)^{-\frac{1}{2}}v(r)$ into the wave map equation and linearizing, we obtain the eigenvalue problem

$$L_n v := (-\partial_{rr} + V_n)v = -\lambda^2 v\,, \qquad V_n(r) = \frac{a^2}{(r^2+a^2)^2} + \ell(\ell+1)\,\frac{\cos(2U_n)}{r^2+a^2}$$

- The operator L_n has no negative eigenvalues. Proof: $v_n = \sqrt{r^2+a^2}\,U_n'(r)$ is the zero mode of the operator $L_n - a^2/(r^2+a^2)^2$.

- Quasinormal modes: eigenmodes satisfying $v(r) \sim \exp(\mp\lambda r)$ for $r \to \pm\infty$ for $\Re(\lambda) < 0$
- $\Re(\lambda)$ tends rapidly to zero as ℓ grows (metastable trapping of waves)



Hyperboloidal initial value problem

- We define new variables ($-\infty < s < \infty$, $-\pi/2 < y < \pi/2$)

$$s = t - \sqrt{r^2 + a^2}, \quad y = \arctan(r/a)$$

- The wave map equation takes the form

$$U_{ss} + 2\sin y\, U_{sy} + \frac{1+\sin^2 y}{\cos y}\, U_s = \cos^2 y\, U_{yy} - \frac{\ell(\ell+1)}{2}\, \sin(2U)\,.$$

- The Bondi-type energy

$$\mathscr{E} = \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left(U_s^2 + \cos^2 y\, U_y^2 + \ell(\ell+1)\, \sin^2 U\right) \frac{dy}{\cos^2 y}$$

- Energy balance ($U(s,y) \sim c_{\pm}(s)\cos(y)$ for $y \to \pm\pi/2$)

$$\frac{d\mathscr{E}}{ds} = -\dot{c}_{-}^2(s) - \dot{c}_{+}^2(s)$$

- Since the energy $\mathscr{E}(s)$ is positive and monotone decreasing, it has a nonnegative limit for $s \to \infty$. It is natural to expect that this limit is given by the energy of a static endstate of evolution.

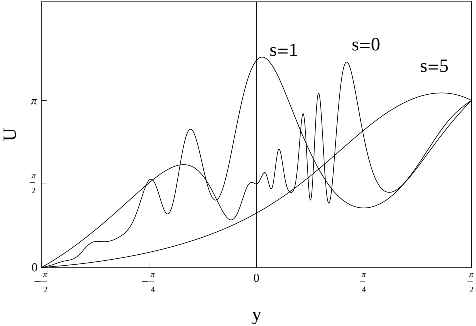
Soliton resolution conjecture

Conjecture

For any smooth initial data of degree n there exists a unique and smooth global solution which converges asymptotically to the harmonic map U_n.

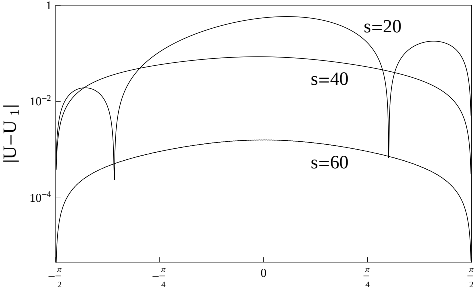
- Note that this is a very strong non-perturbative assertion
- Recently, an analogous result was proved for equivariant wave maps exterior to a ball by Kenig, Lawrie, and Schlag. It seems feasible that their proof can be adopted to our case.
- We believe that the hyperboloidal approach is much better suited for studying this and similar problems. Key advantages:
  - dissipation of energy by radiation through null infinity is inherently incorporated in this formulation
  - *pointwise* convergence to the attractor on the entire spatial domain
  - numerical analysis is relatively easy thanks to spatial compactification and no boundary conditions at the endpoints

Numerical evidence - snaphots from the evolution



'Generic' initial data for $n = 1$ and $\ell = 1$.

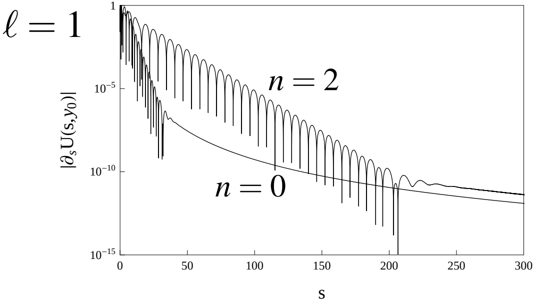
Early times



Late times

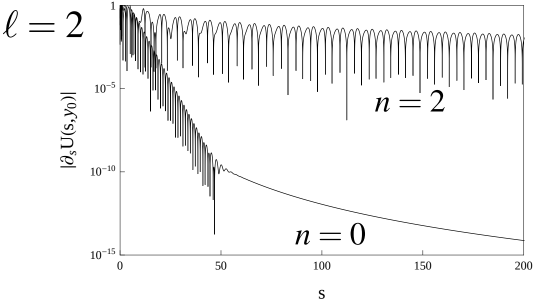
$$U(s,y) - U_1(y) \sim \frac{A_1^{\pm} \cos y (s \cos y + 1)}{s^3 (s \cos y + 2)^3}$$

Numerical evidence - pointwise convergence



Quasinormal modes:
$\lambda_0 = -0.53 + 1.57i$
$\lambda_2 = -0.11 + 0.51i$

Tail $\sim s^{-5}$



Quasinormal modes:
$\lambda_0 = -0.51 + 2.55i$
$\lambda_2 = -0.013 + 0.68i$

Tail $\sim s^{-6}$

Final remarks

- Playing with domains of nonlinear wave equations one can construct toy models for studying interesting physical phenomena (”designer” PDEs)
- The hyperboloidal approach to the initial value problem is ideally suited for studying the relaxation processes due to dispersive dissipation of energy
- Can one turn the hyperboloidal flow method into the proof?