New evidence for the instability of AdS

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Outline

- Brief reminder about AdS
- Nonlinear waves in confined geometries
- Resonant approximation
- New evidence for the instability of AdS
- Conclusions

Anti-de Sitter (AdS) spacetime in $d+1$ dimensions

- AdS is the maximally symmetric solution of the vacuum Einstein equations $R_{\alpha\beta} = \lambda g_{\alpha\beta}$ with negative λ

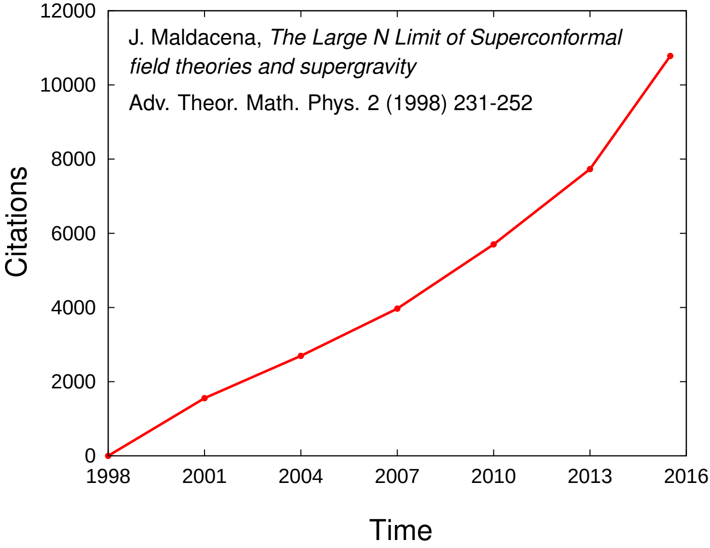
$$g = -(1+r^2/\ell^2)dt^2 + \frac{dr^2}{1+r^2/\ell^2} + r^2 d\Omega^2_{S^{d-1}}$$

  where $\ell^2 = -d/\lambda$, $r \geq 0$, and $-\infty < t < \infty$.

- Substituting $r = \ell \tan x$ ($0 \leq x < \pi/2$) we get

$$g = \frac{\ell^2}{\cos^2 x}\left(-dt^2 + dx^2 + \sin^2 x\, d\Omega^2_{S^{d-1}}\right)$$

- Conformal infinity $x = \pi/2$ is the timelike cylinder $\mathscr{I} = \mathbb{R} \times S^{d-1}$
- Null geodesics reach $\mathscr{I}$ in finite time so AdS is effectively bounded


J. Maldacena, *The Large N Limit of Superconformal field theories and supergravity*
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Citations
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Is AdS stable?

- By the positive energy theorem AdS space is the unique ground state among asymptotically AdS spacetimes (much as Minkowski space is the unique ground state among asymptotically flat spacetimes)
- Minkowski spacetime was proved to be asymptotically stable by Christodoulou and Klainerman (1993)
- Key difference between Minkowski and AdS: **the main mechanism of stability of Minkowski - dissipation of energy by dispersion - is absent in AdS** (for no flux boundary conditions $\mathscr{I}$ acts as a mirror)
- The problem of stability of AdS has not been explored until recently; notable exception: proof of local well-posedness by Friedrich (1995)
- The problem seems tractable only in spherical symmetry so one needs to add matter to generate dynamics. Simple choice: a massless scalar field

- Convenient parametrization of 5D asymptotically AdS spacetimes

$$ds^2 = \frac{\ell^2}{\cos^2 x}\left(-Ae^{-2\delta}dt^2 + A^{-1}dx^2 + \sin^2 x\, d\omega^2\right)$$

  where A and δ are functions of (t,x).
- Define $m(t,x) = \sin^2 x \sec^4 x\,(1-A)$, $\Phi = \partial_x\phi$ and $\Pi = A^{-1}e^{\delta}\partial_t\phi$
- Field equations (using $8\pi G = 3$)

$$\partial_t\Phi = \partial_x\left(Ae^{-\delta}\Pi\right), \quad \partial_t\Pi = \frac{1}{\tan^3 x}\partial_x\left(\tan^3 x Ae^{-\delta}\Phi\right),$$
$$\partial_x m = \tan^3 x A\left(\Phi^2 + \Pi^2\right), \quad \partial_x\delta = -\sin x\cos x\left(\Phi^2 + \Pi^2\right)$$

- Initial-boundary problem is locally well-posed under the following boundary conditions near $x = \pi/2$ (Holzegel-Smulevici 2011)

$$\phi \sim (\pi/2 - x)^4, \quad \delta \sim (\pi/2 - x)^8, \quad 1 - A = (\pi/2 - x)^4$$

- We consider small perturbations of AdS space $\phi = 0, m = 0, \delta = 0$.

AdS gravity with a spherically symmetric scalar field

Conjecture (B-Rostworowski 2011)

AdS_{d+1} (for $d \geq 3$) is unstable under arbitrarily small perturbations

Arguments:

- The linear spectrum is fully resonant. Nonlinear interactions between harmonics give rise to transfer of energy from low to high frequencies.
- The turbulent cascade leads to concentration of energy on finer and finer spatial scales so eventually a black hole is expected to form.
- Numerical evidence: perturbations of size ε collapse in time $\mathcal{O}(\varepsilon^{-2})$.

The shadow of a doubt: is extrapolation to $\varepsilon \to 0$ justified?

New argument (this talk):

- In the limit $\varepsilon \to 0$ we construct an approximate solution that becomes singular in time $\mathcal{O}(\varepsilon^{-2})$.
- This result hints at a possible route to proving the conjecture.

Nonlinear waves in confined geometries

- Consider a nonlinear wave equation for $\phi(t,x)$ with $(t,x) \in R \times M$, where M is a compact Riemannian manifold with metric g.
- Example: $\phi_{tt} - \Delta_g \phi + \phi^3 = 0$ for $M = T^d$ or S^d.
- Goal: understand out-of-equilibrium dynamics of small solutions.
- Due to the lack of dispersion the long-time dynamics is much more complex and mathematically challenging than in the non-compact setting.
- Is the ground state $\phi = 0$ stable (say in H_2 norm)?
- This is an open problem even for $\phi_{tt} - \phi_{xx} + \phi^3 = 0$ on S^1 **!**
- Key enemy: **wave turbulence** - transfer of energy to progressively smaller scales causing gradual loss of smoothness as $t \to \infty$.

Example: $\Box_g \phi - \phi^3 = 0$ on AdS$_5$

$$\partial_{tt}\phi + L\phi + \sec^2 x\, \phi^3 = 0, \quad L = -\tan^{-3} x\, \partial_x (\tan^3 x\, \partial_x) \quad (\star)$$

- Linear spectrum: $L e_n = \omega_n^2 e_n$ where $\omega_n^2 = (2n+4)^2$ ($n = 0, 1, \dots$)
- Plugging the mode expansion $\phi(t,x) = \sum_n c_n(t) e_n(x)$ into ($\star$) we get

$$\frac{d^2 c_n}{dt^2} + \omega_n^2 c_n = \sum_{jkl} I_{jkln}\, c_j c_k c_l, \quad I_{jkln} = -\int_0^{\pi/2} e_j(x) e_k(x) e_l(x) e_n(x) \sin^3 x \cos x\, dx$$

- In the interaction picture, defined by variation of constants,

$$c_n = \beta_n e^{i\omega_n t} + \bar{\beta}_n e^{-i\omega_n t}, \quad \frac{dc_n}{dt} = i\omega_n \left(\beta_n e^{i\omega_n t} - \bar{\beta}_n e^{-i\omega_n t}\right)$$

  this becomes

$$2i\omega_n \frac{d\beta_n}{dt} = \sum_{jkl} I_{jkln}\, c_j c_k c_l\, e^{-i\omega_n t}$$

- Each term in the sum has a factor $e^{-i\Omega t}$, where $\Omega = \omega_n \pm \omega_j \pm \omega_k \pm \omega_l$. Two kinds of terms: $\Omega = 0$ (**resonant**) and $\Omega \neq 0$ (non-resonant).

Resonant approximation

- We define the slow time $\tau = \varepsilon^2 t$ and rescale $\beta_n(t) = \varepsilon\alpha_n(\tau)$.
- The non-resonant terms $\propto e^{-i\Omega\tau/\varepsilon^2}$ are highly oscillatory for small ε and therefore negligible (at least for some time).
- Keeping only the resonant terms (which is equivalent to time-averaging), we obtain the infinite autonomous dynamical system (**resonant system**)

$$2i\omega_n \frac{d\alpha_n}{d\tau} = \sum_{jkl} I_{jkln}\, \alpha_j \alpha_k \bar{\alpha}_l \,,$$

  where the summation runs over the set of indices $\{jkl\}$ for which $\Omega = 0$ and $I_{jkln} \neq 0$. This set can be shown to reduce to $\{jkl \,|\, j+k-l=n\}$.
- The resonant system is invariant under the scaling $\alpha_n(\tau) \to \varepsilon^{-1}\alpha_n(\tau/\varepsilon^2)$
- The resonant approximation is valid on the timescale $\mathcal{O}(\varepsilon^{-2})$. Thus, on this timescale the dynamics of solutions of the cubic wave equation is dominated by resonant interactions.

Resonant approximation for the AdS Einstein-scalar system

- At the lowest order the resonant system has the same form as above

$$2i\omega_n \frac{d\alpha_n}{d\tau} = \sum_{j+k-l=n} C_{jkln}\, \alpha_j \alpha_k \bar{\alpha}_l\,, \quad (RS)$$

  but the interaction coefficients C_{jkln} are *very* complicated.

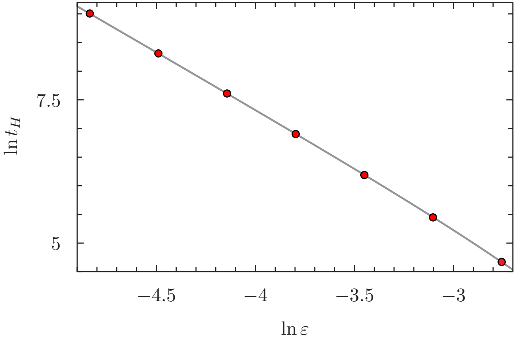
- (RS) was derived by Balasubramanian-Buchel-Green-Lehner-Liebling and Craps-Evnin-Vanhoof (2014) using the multiscale perturbation methods and the averaging method.

- Remark: in the multiscale approach (RS) follows from elimination of secular terms due to resonances at the third order of perturbation expansion. The fact that all secular terms can be removed in this way has been sometimes misunderstood as evidence for stability of AdS.

- We shall analyze (RS) using numerical and asymptotic methods.

Full GR evolution

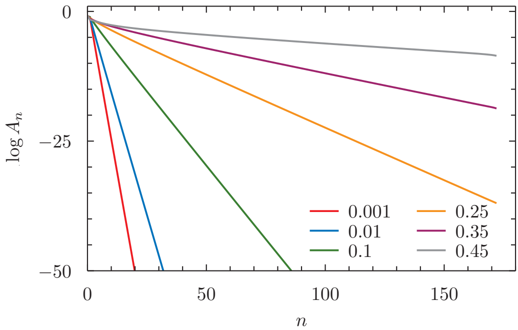
- We shall illustrate the numerical results using the time-symmetric two-mode initial data $\phi(0,x) = \varepsilon\left(\frac{1}{4}e_0(x) + \frac{1}{6}e_1(x)\right)$
- Key observation (B-Rostworowski 2011): horizon forms in time $t_H(\varepsilon) \sim \varepsilon^{-2}$



- This scaling suggests that the instability of AdS should be seen in the resonant approximation.

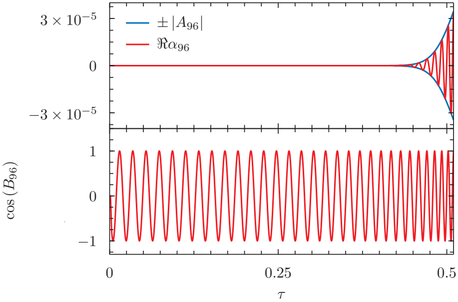
Truncated resonant approximation - numerics

- For the numerical computation we truncate (RS) at $N = 172$ (TRS)
- It is convenient to use the amplitude-phase representation $\alpha_n = A_n e^{iB_n}$
- For the two-mode initial data the higher modes are quickly excited



- For early times $A_n(\tau) \sim \tau^{n-1}$ while the phases $B_n(\tau)$ evolve linearly.

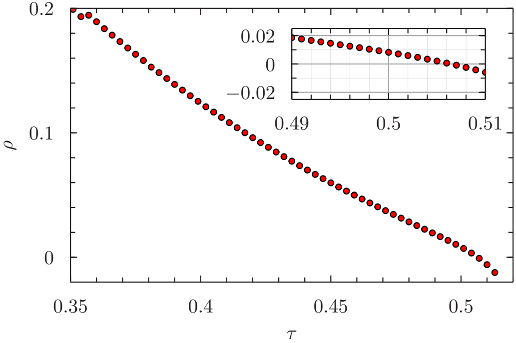
Later times



- A highly oscillatory behavior develops causing numerical difficulties.
- The time-step of numerical integration, for which the algorithm is convergent, tends to zero as the cutoff N increases.
- This suggests that the solution of (RS) develops an oscillatory singularity in some finite time τ_*.
- Remark: for any finite N the solution of TRS can be numerically continued past τ_*, however this ‘afterlife’ is an artifact of truncation.

Analyticity strip method (Sulem-Sulem-Frisch 1983)

- We make the ansatz $A_n(\tau) \sim n^{-\gamma(\tau)} e^{-\rho(\tau)n}$ for large n.
- Fitting to the data we get



- It appears that the 'analyticity radius' $\rho(\tau)$ tends to zero in a finite time τ_*.
- Moreover, the fit reveals that $\lim_{\tau\to\tau_*} \gamma(\tau) = 2$.

Resonant system

- Substituting $\alpha_n = A_n e^{iB_n}$ into (RS) one gets

$$2\omega_n \frac{dA_n}{d\tau} = \sum_{\substack{j+k-l=n \\ j\neq n, k\neq n}} S_{jkln} A_j A_k A_l \sin(B_n + B_l - B_j - B_k)$$

$$2\omega_n \frac{dB_n}{d\tau} = T_n A_n^2 + \sum_{j\neq n} R_{jn} A_j^2 + A_n^{-1} \sum_{\substack{j+k-l=n \\ j\neq n, k\neq n}} S_{jkln} A_j A_k A_l \cos(B_n + B_l - B_j - B_k)$$

- Explicit expressions for the coefficients T_n, R_{jn}, and S_{jkln} were derived by Craps-Evnin-Vanhoof (a remarkable tour de force calculation!)
- We will argue that:
  - the resonant system has a solution that becomes singular in finite time
  - this singular solution governs the generic blowup

Asymptotic analysis

- We assume that $A_n(\tau) \sim n^{-2} e^{-\rho_0(\tau_* - \tau)n}$ for large n and $\tau \to \tau_*$
- Asymptotic behavior of the interaction coefficients

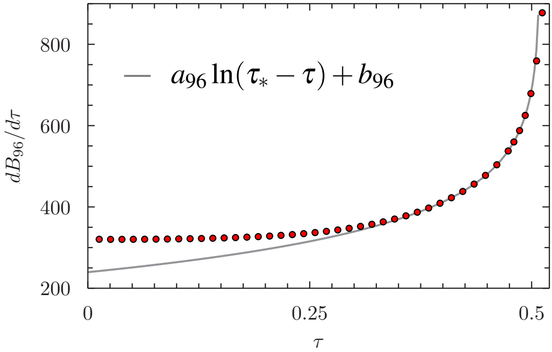
$$T_n \sim n^5, \quad R_{jn} \sim n^2 j^3, \quad S_{\lambda j, \lambda k, \lambda l, \lambda n} \sim \lambda^4 S_{jkln}$$

- The latter implies that $\sum_{\substack{j+k-l=n \\ j \neq n, k \neq n}} S_{jkln} (jkl)^{-2} = \mathscr{O}(1)$
- It follows that for $\tau \to \tau_*$

$$\sum R_{jn} A_j^2 \sim n^2 \sum j^{-1} e^{-2\rho_0(\tau_* - \tau)j} \sim n^2 \ln(\tau_* - \tau)$$

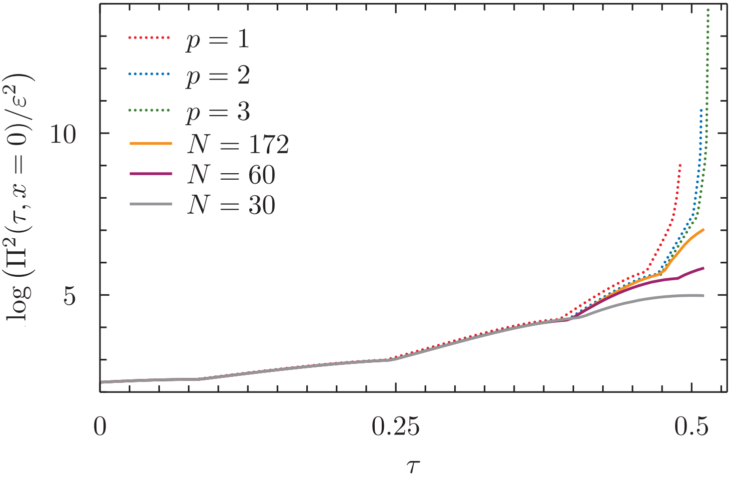
- $\dfrac{dB_n}{d\tau}$ **blows up logarithmically**
- Moreover, B_n behave linearly with n, hence $B_n + B_l - B_j - B_k \approx 0$ for the resonant quartets (confirming that the ansatz is self-consistent).

Numerical confirmation

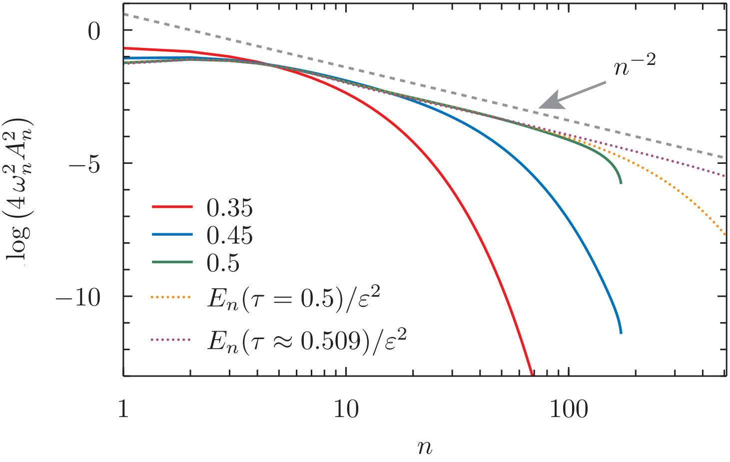


Performing this fit for all $n > 20$ we confirm that the coefficients a_n and b_n vary linearly with n, while $\tau_* \approx 0.509$ does not depend on n. The blowup time τ_* is close to the collapse time for the true solution $\tau_H := \lim_{\varepsilon\to 0} \varepsilon^{-2} t_H(\varepsilon) \approx 0.514$.

How good is the resonant approximation?



Energy spectrum



Conclusions

- Dynamics of asymptotically AdS spacetimes is an interesting meeting point of basic problems in general relativity and PDE theory.
- Understanding of out-of-equilibrium dynamics of small solutions is mathematically challenging even for the simplest nonlinear wave equations on compact manifolds, let alone Einstein's equations.
- For AdS-Einstein-scalar equations, we have constructed the asymptotic solution of the resonant system that becomes singular in finite time. Numerics shows that this solution acts as a universal attractor for blowup.
- Key question: how to transfer this blowup result from the resonant system to the full system? It is not clear to us what (if any) is the physical interpretation of the oscillatory singularity for the resonant system.
- Nonetheless, the fact that solutions of the resonant system blow up in finite time (for typical initial data) strongly indicates that the corresponding solutions of the full system collapse on the timescale $\mathcal{O}(\varepsilon^{-2})$.