Sine-Gordon on a Wormhole

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Cambridge, 6 March 2018

Soliton resolution conjecture

Solutions of dispersive wave equations asymptotically resolve into a superposition of a coherent structure (soliton, black hole,...) and radiation.

- Mathematical understanding of this conjecture is rather limited, especially in the non-perturbative regime (for initial data far from an equilibrium).
- The simplest setting for studying this conjecture:
  1. evolution is globally regular
  2. there is a unique stable (nontrivial) stationary equilibrium
  3. the equilibrium solution is rigid
  4. there are no internal oscillation modes
- Such a model (wave map on a wormhole) was proposed by B-Kahl (2016) and later the soliton resolution conjecture was proven by Rodriguez
- In this talk we consider a similar model but without the property 4

Stability of kink in the ϕ^4 model

- For $\phi : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$

$$\phi_{tt} = \phi_{xx} + \phi(1 - \phi^2)$$

- Kink: $\phi(t,x) = H(x) = \tanh(x/\sqrt{2})$
- Is the kink asymptotically stable?
- Let $\phi(t,x) = H(x) + u(t,x)$. Then $u_{tt} + Lu + f(u,x) = 0$, where

$$L = -\partial_{xx} + V(x) + 2, \quad V(x) = 3H^2(x) - 3 = -3\,\mathrm{sech}^2(x/\sqrt{2})$$

- Spectrum: $\sigma(L) = \{0, 3/2\} \cup [2, +\infty)$
- Key difficulties: internal mode and slow dispersive decay
- For odd perturbations Kowalczyk-Martel-Muñoz (2017) proved that $\lim_{t\to\pm\infty} \|u(t)\|_E = 0$ but the decay estimate is not optimal (which is conjectured to be $t^{-1/2}$; see e.g. Manton-Merabet 1996)

Sine-Gordon equation

- For $\phi : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$

$$\phi_{tt} = \phi_{xx} - \sin(2\phi)$$

- Sine-Gordon equation is completely integrable
- Kink: $\phi(t,x) = H(x) = 2\arctan\left(e^{\sqrt{2}x}\right)$
- Let $\phi(t,x) = H(x) + u(t,x)$. Then $u_{tt} + Lu + f(u,x) = 0$, where

$$L = -\partial_{xx} + V(x) + 2, \quad V(x) = -4\sin^2 H(x) = -4\,\mathrm{sech}^2(\sqrt{2}x)$$

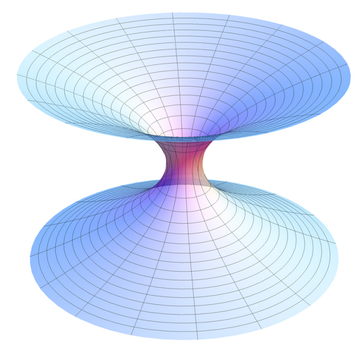
- Spectrum: $\sigma(L) = \{0\} \cup [2, +\infty)$
- Absence of internal modes seems to be intimately tied with integrability
- In the neighbourhood of the SG kink there exists a one-parameter family of wobbling kinks, therefore the SG kink is not asymptotically stable
- Here we shall consider a non-integrable deformation of SG equation

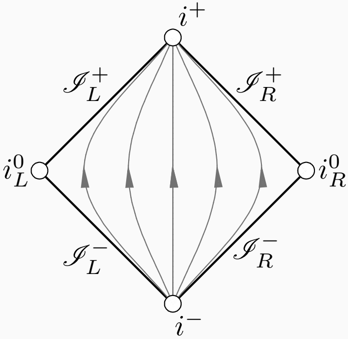
Wormhole

- Domain $M = \{t \in \mathbb{R}, (r, \vartheta, \varphi) \in \mathbb{R} \times \mathbb{S}^2\}$ with metric

$$g = -dt^2 + dr^2 + (r^2 + a^2)(d\vartheta^2 + \sin^2\vartheta d\varphi^2)$$

- Hypersurfaces $t =$ const have two asymptotically flat ends at $r \to \pm\infty$ connected by a neck of area $4\pi a^2$ at $r = 0$.





Sine-Gordon on the wormhole: $\Box_g \phi = \sin(2\phi)$

$$\phi_{tt} = \phi_{rr} + \frac{2r}{r^2+a^2}\,\phi_r - \sin(2\phi)$$

- The length scale a plays two roles:
  - removes the singularity at $r=0 \Rightarrow$ global-in-time regularity
  - breaks scale invariance $\Rightarrow$ allows for kinks
- The equation is truly $1+1$ dimensional ($-\infty < r < \infty$), yet it inherits strong dispersive decay from the original $3+1$ dimensional problem.
- Conserved energy

$$E(\phi) = \int_{-\infty}^{\infty} \left(\frac{1}{2}\phi_t^2 + \frac{1}{2}\phi_r^2 + \sin^2\phi\right)(r^2+a^2)\,dr$$

- Finiteness of energy requires that $\phi(t,-\infty) = m\pi$, $u(t,\infty) = n\pi$. We choose $m=0$ so n determines the topological degree of the map.

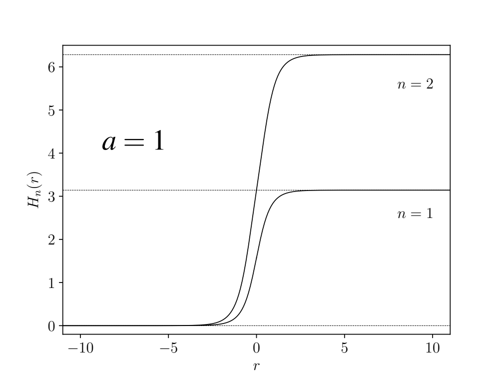
Kinks

$$H'' + \frac{2r}{r^2+a^2} H' - \sin(2H) = 0$$

Theorem

For any given $a > 0$ there exists a unique smooth solution H_n of degree n.

Proof: elementary shooting argument



Analytic approximations for kinks

- For $n = 1$ and large a

$$H_1(r) \approx H_{SG}(r) + \frac{1}{a^2} h_1(r), \qquad h_1'' - 2\cos(2H_{SG})h_1 = -2rH_{SG}'$$

- For $n \geq 2$ and large a the kink $H_n(r)$ is well approximated by a superposition of n sine-Gordon kinks. For example

$$H_2(r) \approx H_{SG}\left(r + \frac{\ln a}{\sqrt{2}}\right) + H_{SG}\left(r - \frac{\ln a}{\sqrt{2}}\right)$$
$$H_3(r) \approx H_{SG}\left(r + \sqrt{2}\ln a\right) + H_{SG}(r) + H_{SG}\left(r - \sqrt{2}\ln a\right)$$

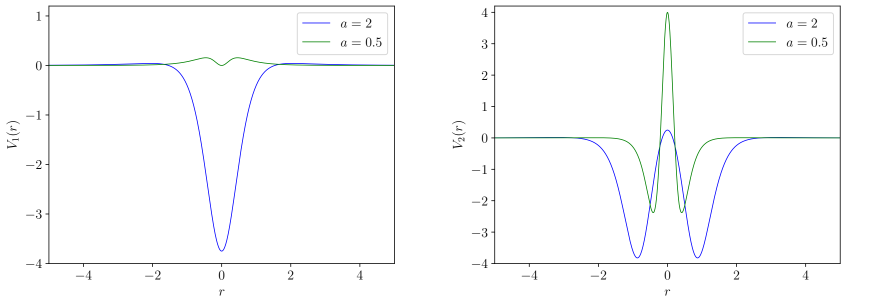
- For small a

$$H_n(r) \approx n\left(\frac{\pi}{2} + \arctan(r/a)\right)$$

Linear perturbations

- Let $\phi(t,r) = H_n(r) + \dfrac{u(t,r)}{\sqrt{r^2+a^2}}$. Then $u_{tt} + L_n u + f(u,r) = 0$, where

$$L_n = -\partial_{rr} + V_n(r) + 2, \qquad V_n(r) = -4\sin^2 H_n(r) + \frac{a^2}{(r^2+a^2)^2}$$



Spectrum of L_n

- Consider the eigenvalue problem $L_1\psi = \omega^2\psi$. For large a

$$\psi = \psi_0 + \frac{1}{a^2}\psi_1 + \mathcal{O}\left(\frac{1}{a^4}\right), \quad \omega^2 = \xi\frac{1}{a^2} + \mathcal{O}\left(\frac{1}{a^4}\right)$$

  where $\psi_0(r) = 2^{-3/4}H'_{SG}(r)$ is the zero mode of sine-Gordon kink

- At order $\mathcal{O}(1/a^2)$ we get

$$\xi = 1 - 4\int_{-\infty}^{\infty}\sin(2H_{SG}(r))h_1(r)\psi_0^2(r)\,dr = 2$$

- As a decreases, the eigenvalue ω^2 migrates through the gap $(0,2)$ and disappears into the continuous spectrum for $a < a^* \approx 0.536$.
- For $n \geq 2$ and sufficiently large a there are n gap eigenvalues. As a decreases they disappear one by one into the continuous spectrum at certain critical values $a_1^* < a_2^* < .... < a_n^*$. For example, for $n = 2$ we find $a_1^* \approx 0.39$ and $a_2^* \approx 0.81$.

Conjecture

For any smooth finite-energy initial data of degree n there exists a unique smooth global solution which converges asymptotically to the kink H_n.

- If there are no internal modes, then $|u(t)| \sim t^{-3/2}$ for $t \to \infty$
- The decay of internal modes is due to their resonant interactions with radiation (Soffer-Weinstein 1999)
- Consider the case $n=1$ and let P_ψ be a projector on the internal mode ψ. Decomposing $u(t,r) = a(t)\psi(r) + \eta(t,r)$ and projecting, we get

$$a_{tt} + \omega^2 a + P_\psi f(a\psi + \eta, r) = 0, \quad \eta_{tt} + L_1\eta + P_\psi^\perp f(a\psi + \eta, r) = 0$$

- Solving for η and substituting $a = Ae^{i\omega t} + \bar{A}e^{-i\omega t}$, we get ($\Gamma > 0$)

$$\partial_t |A| \simeq -\Gamma |A|^{2N+1}, \qquad N\omega < \sqrt{2} < (N+1)\omega$$

hence $|A| \sim t^{-\frac{1}{2N}}$ for $t \to \infty$

Hyperboloidal initial value problem

- We define new "hyperboloidal" coordinates

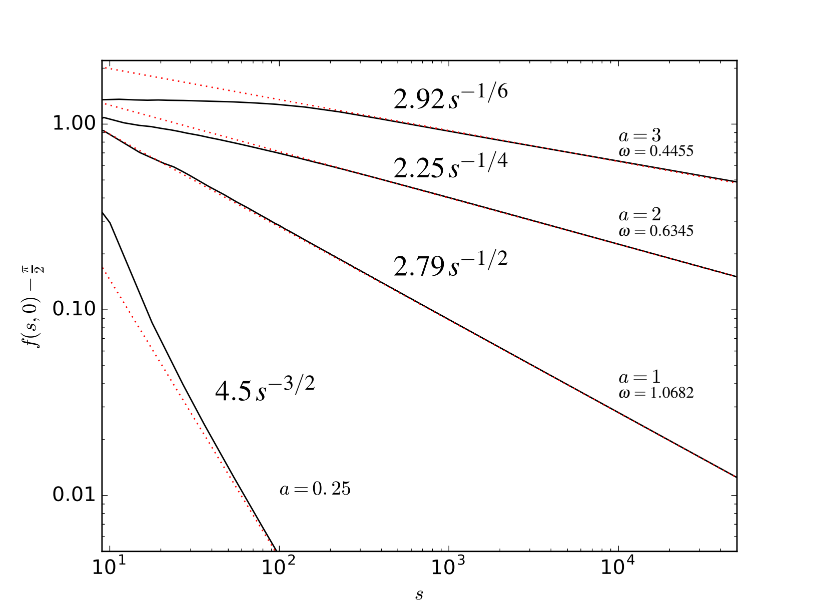
$$s = \frac{t}{a} - \sqrt{\frac{r^2}{a^2} + 1}, \quad y = \arctan\left(\frac{r}{a}\right)$$

- Our equation for $f(s, y) = \phi(t, r)$ takes the form

$$f_{ss} + 2 \sin y f_{sy} + \frac{1 + \sin^2 y}{\cos y} f_s = \cos^2 y f_{yy} - a^2 \frac{\sin(2f)}{\cos^2 y} \qquad (\star)$$

- There are no ingoing characteristics at the boundaries, hence no boundary conditions are required (or allowed).
- We solve equation $(\star)$ for smooth initial data $\phi(0, y)$, $\phi_s(0, y)$ that are compactly supported perturbations of the kink.

Numerical evidence



Numerical evidence