Dimensional analysis
(physics made easy)

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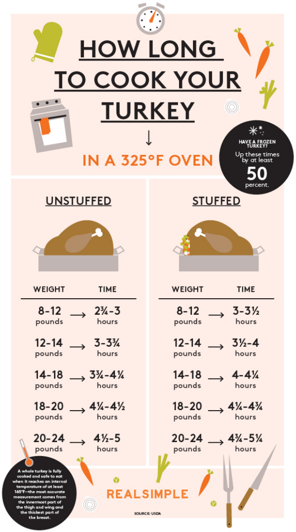


$$t = c\,m^{\frac{2}{3}}$$

↓

constant determined from a single measurement

It happens not infrequently that results in the form of 'laws' are put forward as novelties on the basis of elaborate experiments, which might have been predicted a priori after a few minutes' consideration..

– Lord Rayleigh, 1915
The principle of similitude

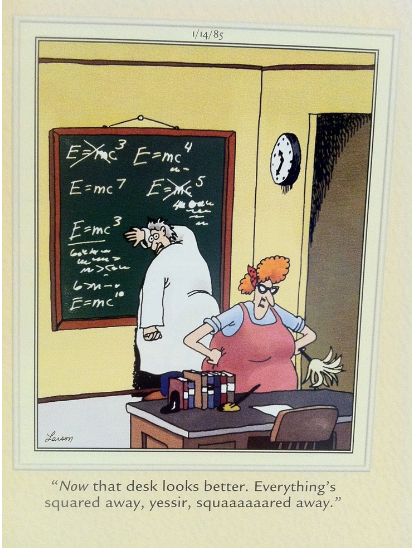
BASIC IDEA OF DIMENSIONAL ANALYSIS

The laws of physics should not depend on the choice of physical units.

$$\Downarrow$$

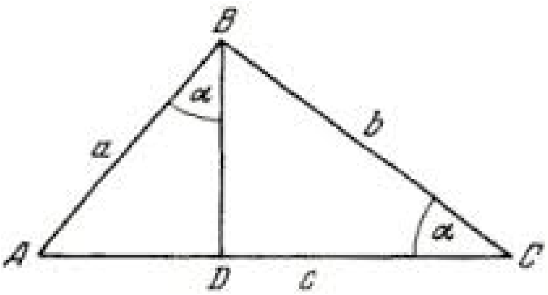
Every law can be written in a dimensionless form.

Check your units!



"*Now* that desk looks better. Everything's squared away, yessir, squaaaaaared away."

Pythagoras theorem

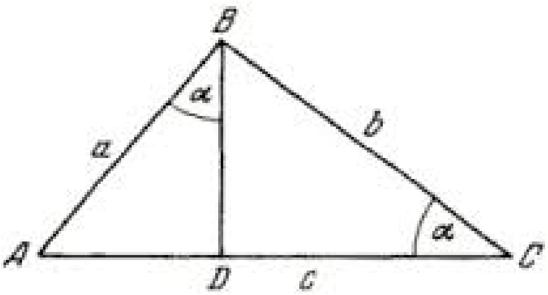


Area of the right triangle $=$ hypotenuse$^2 \times f(\alpha)$

$$a^2 f(\alpha) + b^2 f(\alpha) = c^2 f(\alpha)$$

This ‘proof’ fails in a non-Euclidean space!

Pythagoras theorem

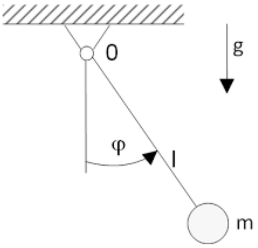


Area of the right triangle = hypotenuse$^2 \times f(\alpha)$

$$a^2 \cancel{f(\alpha)} + b^2 \cancel{f(\alpha)} = c^2 \cancel{f(\alpha)}$$

This ‘proof’ fails in a non-Euclidean space!

Pendulum



Variables:

- frequency $[\omega] = T^{-1}$
- mass $[m] = M$
- length $[l] = L$
- gravity $[g] = LT^{-2}$
- angle $[\phi] = 1$

- Two dimensionless quantities: $\frac{\omega^2 l}{g}$ and ϕ
- Thus, $\frac{\omega^2 l}{g} = f(\phi) \implies \boxed{\omega^2 = \frac{g}{l} f(\phi)}$
- For small angles

$$\omega^2 \approx \frac{g}{l} f(0)$$

assuming that $\lim_{x \to 0} f(x) \neq 0$ exists.

Dimensional analysis - instruction manual

1. Make a list of n physical variables relevant for the problem
   ($n = 5$ for the pendulum: ω, m, l, g, ϕ)
2. Establish the number of independent dimensions k
   ($k = 3$ for the pendulum: L, M, T)
3. Write down $n - k$ dimensionless combinations $\alpha_1, \ldots, \alpha_{n-k}$
   ($n - k = 2$ for the pendulum: $\alpha_1 = \omega^2 l/g$, $\alpha_2 = \phi$)
4. The solution of your problem can be written in the form

$$F(\alpha_1, \ldots, \alpha_{n-k}) = 0$$

5. Use physics (intuition) to get rid of irrelevant quantities
   (consider limiting cases $\alpha_j \to 0$ or $\alpha_j \to \infty$)

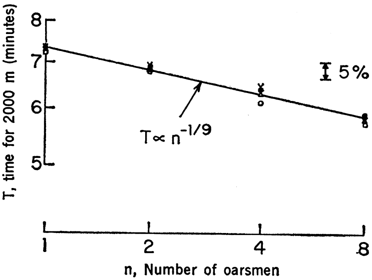
Terminal velocity

Variables:

- drag $[F] = MLT^{-2}$
- size $[R] = L$,
- density of air $[\rho] = ML^{-3}$
- velocity $[\upsilon] = LT^{-1}$
- viscosity $[\mu] = ML^{-1}T^{-1}$

- Dimensionless quantities: $\alpha_1 = \frac{\rho\upsilon R}{\mu}$ (Reynolds number), $\alpha_2 = \frac{F}{\rho R^2\upsilon^2}$
- Thus $\frac{F}{\rho R^2\upsilon^2} = f(\alpha_1) \Longrightarrow \boxed{F = \rho R^2\upsilon^2 f(\alpha_1)}$
- Assuming that $\lim_{\alpha_1\to\infty} f(\alpha_1) \neq 0$ exists, we get $\boxed{F \sim \rho R^2\upsilon^2}$ for large α_1
- Balance between the gravitational force and the drag $\rho R^2\upsilon^2 \sim mg = \rho_0 R^3 g$ gives the terminal velocity $\upsilon \sim \left(\frac{gR\rho_0}{\rho}\right)^{1/2}$

Speed of the rowing boats



McMahon (1971)

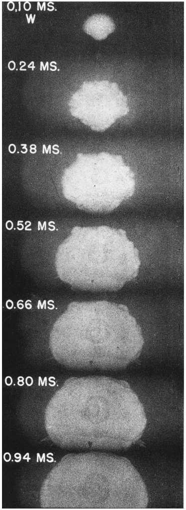
- Drag $F \sim \rho \upsilon^2 \ell^2$
- Submerged volume $\sim \ell^3 \sim N$
- Power $P \sim N$
- From $F\upsilon = P$ it follows

$$\boxed{\upsilon \sim N^{1/9}}$$

- For coxed boat $\ell^3 \sim (N + 1/2)$

$$\upsilon \sim \frac{N^{1/3}}{(N+1/2)^{2/3}}$$

Speed of the shock wave



Life Magazine 1945

- Radius $r = f(E, \rho, t, \ldots)$
- Dimensionless quantity $\dfrac{Et^2}{\rho r^5}$
- Thus $\boxed{r = c\left(\dfrac{E}{\rho}\right)^{\frac{1}{5}} t^{2/5}}$

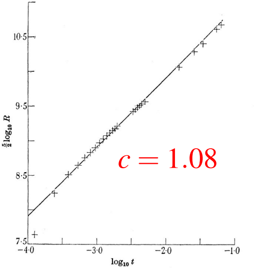
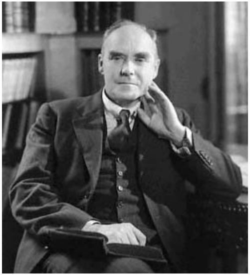


FIGURE 1. Logarithmic plot showing that $R^{\frac{5}{2}}$ is proportional to t.



G.I. Taylor

Cooking the turkey

Heat equation $\partial_t u = \kappa \Delta u$

- temperature inside the turkey $[u] = K$
- temperature in the oven $[u_0] = K$
- mass $[m] = M$
- density $[\rho] = ML^{-3}$
- time $[t] = T$

- Dimensionless quantities: $\dfrac{u}{u_0}$ and $\dfrac{\rho(\kappa t)^{3/2}}{m}$
- $\dfrac{u}{u_0} = f\left(\dfrac{\rho(\kappa t)^{3/2}}{m}\right) \Longrightarrow \boxed{t \sim m^{2/3}}$
- Note that K is treated as an independent dimension!

Limitations of dimensional analysis

- Speed of water waves

$$v^2 = \frac{g\lambda}{2\pi} \tanh\left(\frac{2\pi h}{\lambda}\right)$$

  - For $h \gg \lambda$ (deep water): $v^2 \sim g\lambda$
  - For $h \ll \lambda$ (shallow water): $v^2 \sim gh$

- Suppose that dimensional analysis predicts that

$$w \sim r^{\alpha} f(h/r),$$

where h and r are two length scales. Let $h \ll r$.

  - If $f(x)$ has a nonzero limit at $x = 0$, then $w \sim r^{\alpha}$
  - If $f(x) \sim x^{\delta}$ for small x, then $w \sim r^{\alpha-\delta}$ (anomalous scaling)