Wave Maps on a Wormhole

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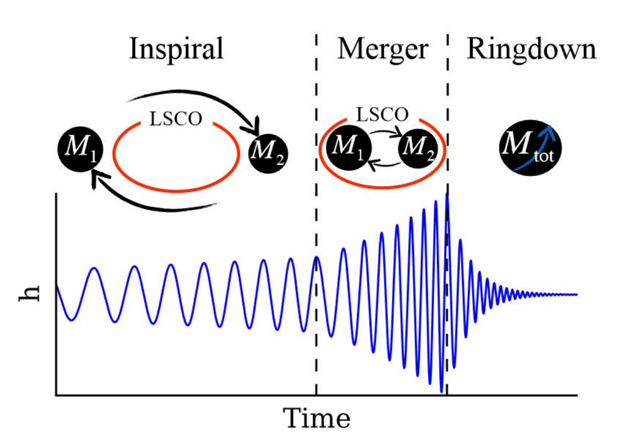
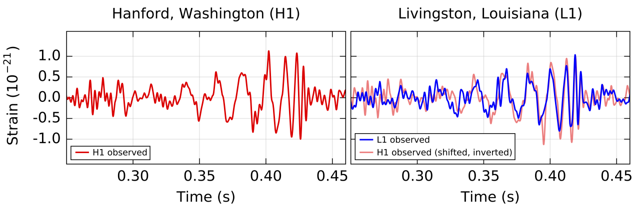
Introduction

- Part of a long-term project on global dynamics of nonlinear wave equations (wave maps, Yang-Mills equations, Einstein equations).
- Our approach is not rigorous - what we do can be described as: *the physics of the mathematics of the physics* (M. Berry)
- Goal: understanding of relaxation to a stationary equilibrium on an unbounded domain through the dissipation of energy by dispersion.
- The emitted radiation encodes information about an attractor.
- Example: the mass and spin of a stationary black hole can be read off from characteristic frequencies of gravitational waves emitted in the process relaxation to the equilibrium (so called ringdown).



Inspiral
Merger
Ringdown
LSCO
LSCO
M_1
M_2
M_1
M_2
M_{tot}
h
Time



Hanford, Washington (H1)
Livingston, Louisiana (L1)
Strain (10^{-21})
1.0
0.5
0.0
-0.5
-1.0
H1 observed
L1 observed
H1 observed (shifted, inverted)
0.30
0.35
0.40
0.45
Time (s)
0.30
0.35
0.40
0.45
Time (s)

Soliton resolution conjecture

Solutions of dispersive wave equations asymptotically resolve into a superposition of a coherent structure (soliton, black hole,...) and radiation.

- Mathematical understanding of this conjecture is rather limited, especially in the nonperturbative regime (for initial data far from the equilibrium).
- We want to design a simple playground for studying this conjecture.
- The desired properties of a toy model:
  - evolution is globally regular
  - there is a unique stable (nontrivial) stationary equilibrium
  - the equilibrium solution is rigid
  - there are no oscillatory modes
- To construct such a model we consider equivariant wave maps into the 3-sphere with a suitably chosen curved manifold as a domain.

Equivariant wave maps

- Let $U : M \mapsto N$ be a map from a Lorentzian manifold (M, g) into a Riemannian manifold (N, G).
- Domain M: ultrastatic spherically symmetric spacetime with the metric

$$g = -dt^2 + dr^2 + R^2(r)\,(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)$$

- Target: $N = \mathbb{S}^3$ with the round metric $G = du^2 + \sin^2 u\,(d\theta^2 + \sin^2\theta\, d\phi^2)$.
- Equivariant ansatz: $u = u(t,r), (\theta, \phi) = \chi_\ell(\theta, \phi)$, where $\chi_\ell : S^2 \mapsto S^2$ is an eigenmap map with eigenvalue $\ell(\ell+1)$ ($\ell \in \mathbb{N}$).
- Let $W_{\alpha\beta} = \partial_\alpha U^A \partial_\beta U^B G_{AB}$. The wave map action is

$$S = \int_M g^{\alpha\beta} W_{\alpha\beta}\, dv_M = 4\pi \int \left(-u_t^2 + u_r^2 + \ell(\ell+1)\,\frac{\sin^2 u}{R^2} \right) R^2\, dr\, dt$$

which gives the equivariant wave map equation

$$u_{tt} = u_{rr} + \frac{2R'(r)}{R^2(r)}\, u_r - \frac{\ell(\ell+1)}{2}\, \frac{\sin(2u)}{R^2(r)}$$

Minkowski background

$$u_{tt} = u_{rr} + \frac{2}{r} u_r - \frac{\ell(\ell+1)}{2} \frac{\sin(2u)}{r^2}$$

- Scaling symmetry $u(t,r) \mapsto u_\lambda(t,r) = u(t/\lambda, r/\lambda)$.
- Supercritical scaling of energy $E(u_\lambda) = \lambda E(u)$.
- No static solutions (harmonic maps) with finite nonzero energy.
- Small data: solutions are global in time and decay (nonlinearly) to zero

$$u(t,r) \sim \frac{A r^\ell}{\langle t-r\rangle^{\ell+1} \langle t+r\rangle^{\ell+1}} \quad \text{for} \quad t \to \infty$$

- Large data: self-similar blowup in finite time

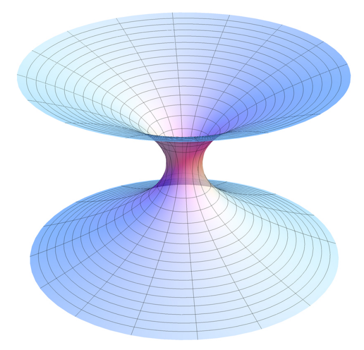
$$u(t,r) \sim S\left(\frac{r}{T-t}\right) \quad \text{for} \quad t \nearrow T$$

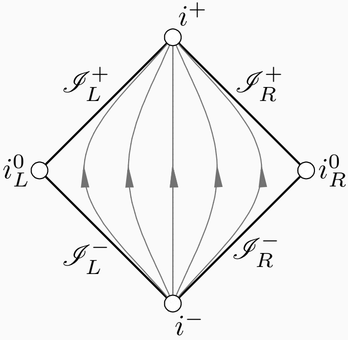
Wormhole background

- Domain $M = \{t \in \mathbb{R}, (r, \vartheta, \varphi) \in \mathbb{R} \times \mathbb{S}^2\}$ with metric

$$g = -dt^2 + dr^2 + (r^2 + a^2)(d\vartheta^2 + \sin^2\vartheta d\varphi^2)$$

- Hypersurfaces $t =$ const have two asymptotically flat ends at $r \to \pm\infty$ connected by a neck of area $4\pi a^2$ at $r = 0$.





Equivariant wave map equation on the wormhole

$$u_{tt} = u_{rr} + \frac{2r}{r^2+a^2}\, u_r - \frac{\ell(\ell+1)}{2}\, \frac{\sin(2u)}{r^2+a^2}$$

- The length scale a plays two roles:
  - removes the singularity at $r = 0 \Rightarrow$ global-in-time regularity
  - breaks scale invariance $\Rightarrow$ allows for harmonic maps
- The equation is truly $1+1$ dimensional ($-\infty < r < \infty$), yet it inherits strong dispersive decay from the original $3+1$ dimensional problem.
- There are close analogies between this equation and the exterior wave map equation (B-Chmaj-Maliborski, Kenig-Lawrie-Liu-Schlag).
- Conserved energy

$$E(u) = \frac{1}{2} \int_{-\infty}^{\infty} \left[u_t^2 + u_r^2 + \frac{\ell(\ell+1)}{r^2+a^2}\, \sin^2 u \right] (r^2+a^2)\, dr$$

- Finiteness of energy requires that $u(t,-\infty) = m\pi$, $u(t,\infty) = n\pi$. We choose $m = 0$ so n determines the topological degree of the map.

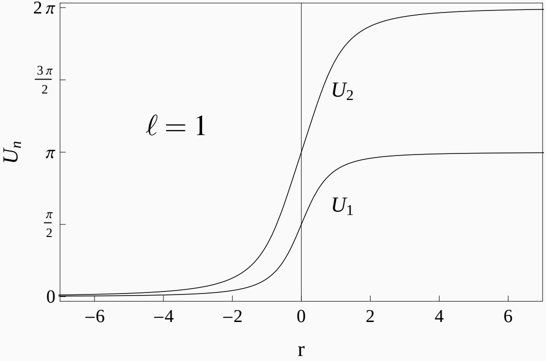
Harmonic maps

$$U'' + \frac{2r}{r^2+a^2}\,U' - \frac{\ell(\ell+1)}{2(r^2+a^2)}\,\sin(2U) = 0\,.$$

Theorem

For any given ℓ there exists a unique smooth harmonic map U_n of degree n.

Proof: elementary shooting argument



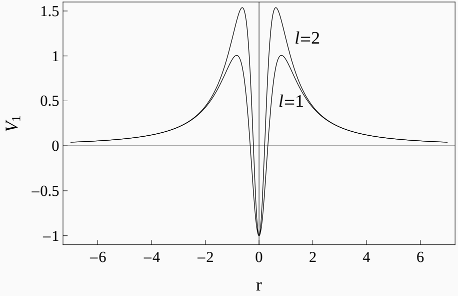
Linear perturbations

- Substituting $u(t,r) = U_n(r) + e^{\lambda t}(r^2+a^2)^{-\frac{1}{2}} v(r)$ into the wave map equation and linearizing, we obtain the eigenvalue problem

$$L_n v := (-\partial_{rr} + V_n) v = -\lambda^2 v\,, \qquad V_n(r) = \frac{a^2}{(r^2+a^2)^2} + \ell(\ell+1)\,\frac{\cos(2U_n)}{r^2+a^2}$$

- The operator L_n has no negative eigenvalues. Proof: $v_n = (r^2+a^2)\,U_n'(r)$ is the zero mode of the operator $L_n - a^2/(r^2+a^2)^2$.

- Quasinormal modes: 'outgoing' solutions $v(r) \sim \exp(\mp\lambda r)$ for $r \to \pm\infty$ and $\Re(\lambda) < 0$
- $\Re(\lambda)$ tends rapidly to zero as ℓ grows (metastable trapping)



Hyperboloidal initial value problem

- We define a new ("hyperboloidal") time coordinate

$$s = t - \sqrt{r^2 + a^2}$$

- The wave map equation takes the form (hereafter, we set $a = 1$)

$$u_{ss} + 2r\sqrt{r^2+1}\,u_{sr} + \frac{2r^2+1}{\sqrt{r^2+1}}\,u_s = \left((r^2+1)u_r\right)_r - \frac{\ell(\ell+1)}{2}\sin(2u)$$

- The associated energy

$$\mathcal{E}(u) = \frac{1}{2}\int_{-\infty}^{\infty}\left(u_s^2 + (r^2+1)\,u_r^2 + \ell(\ell+1)\sin^2 u\right)dr$$

is radiated away through null infinities

$$\frac{d\mathcal{E}}{ds} = -\dot{c}_-^2(s) - \dot{c}_+^2(s)\,,$$

where

$$c_- = \lim_{r\to-\infty} ru(s,r), \quad c_+ = \lim_{r\to\infty} r(u(s,r) - n\pi)$$

are the radiation coefficients.

Hyperboloidal soliton resolution conjecture

- The energy $\mathscr{E}(s)$ is positive and non-increasing, hence it has a limit for $s \to \infty$. It is natural to expect that this limit is given by the energy of the harmonic map.

Conjecture

For any smooth initial data of degree n there exists a unique smooth global solution which converges asymptotically to the harmonic map U_n.

- An analogous result was proven for equivariant wave maps exterior to a ball by Kenig-Lawrie-Liu-Schlag (without the rate of convergence).
- The hyperboloidal approach (due to Friedrich and Zenginoğlu) seems ideally suited for this purpose. Key advantages:
  - dissipation of energy by radiation through null infinity is inherently incorporated in this formulation
  - *pointwise* convergence to the attractor on the entire spatial domain
  - easy to implement numerically (no artificial boundary conditions)

  However, the evolution is only semi-global in space and forward in time.

Numerical solutions

- We compactify the spatial domain $-\infty < r < \infty$ to the interval $[-\pi/2, \pi/2]$ by the transformation $y = \arctan(r)$. Then

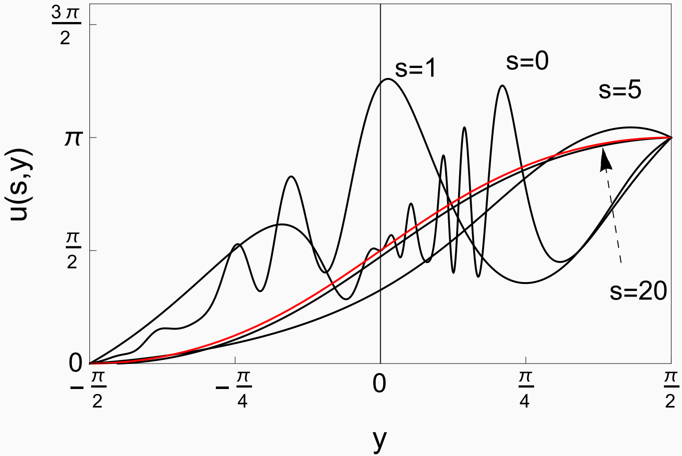
$$u_{ss} + 2\sin y\, u_{sy} + \frac{1+\sin^2 y}{\cos y}\, u_s = \cos^2 y\, u_{yy} - \frac{\ell(\ell+1)}{2}\sin(2u) \qquad (\star)$$

- There are no ingoing characteristics at the boundaries, hence no boundary conditions are required (or allowed).
- We solve equation ($\star$) for smooth initial data $u(0,y)$, $u_s(0,y)$.
- Quasinormal modes are honest eigenfunctions of the quadratic eigenvalue problem

$$(A_n + \lambda B + \lambda^2 I)\, v = 0\,,$$

$$A = -\cos^2 y\, \partial_{yy} + \ell(\ell+1)\cos(2U_n), \quad B = 2\sin y\, \partial_y + \frac{1+\sin^2 y}{\cos y}$$

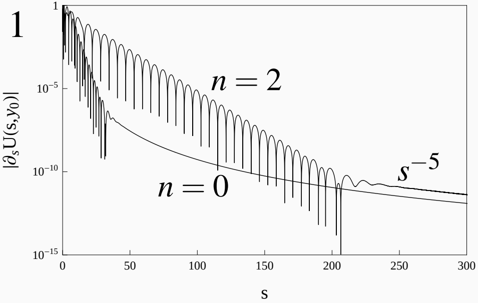
Numerical evidence



Snaphots from the evolution of ‘generic’ initial data for $n = 1$ and $\ell = 1$.

Numerical evidence - rate of convergence

$\ell = 1$



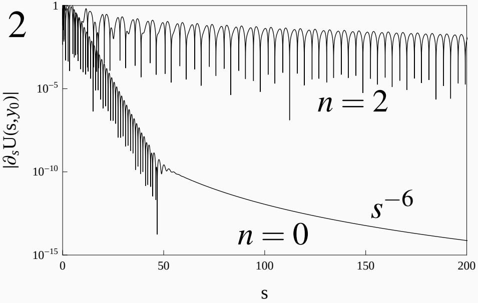
Quasinormal modes:

$\lambda_0 = -0.53 + 1.57i$

$\lambda_2 = -0.11 + 0.51i$

$c_\pm(s) \sim s^{-3}$

$\ell = 2$



Quasinormal modes:

$\lambda_0 = -0.51 + 2.55i$

$\lambda_2 = -0.013 + 0.68i$

$c_\pm(s) \sim s^{-3}$

Conclusion

- The hyperboloidal approach to the initial value problem is ideally suited for studying the relaxation processes due to dispersive dissipation of energy.
- I hope that this approach will attract more attention in the PDE community.