Nonlinear evolution of perturbed anti-de Sitter spacetime

Piotr Bizoń and Andrzej Rostworowski

Jagiellonian University
Kraków

joint work with Dominika Hunik, Joanna Jałmużna, Maciej Maliborski

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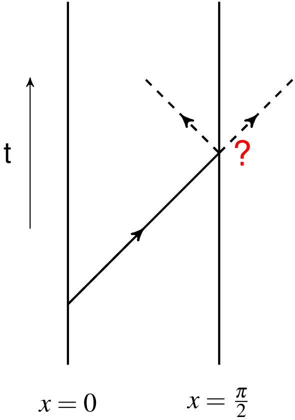
Anti-de Sitter spacetime in $d+1$ dimensions

Manifold $\mathscr{M} = \{t \in \mathbb{R}, x \in [0, \pi/2), \omega \in S^{d-1}\}$ with metric

$$g = \frac{L^2}{\cos^2 x}\left(-dt^2 + dx^2 + \sin^2 x \, d\omega^2_{S^{d-1}}\right)$$

Solution of vacuum Einstein's equations $R_{\alpha\beta} = \lambda g_{\alpha\beta}$ with $\lambda = -d/L^2$.

- Spatial infinity $x = \pi/2$ is the timelike cylinder $\mathscr{I} = \mathbb{R} \times S^{d-1}$ with the boundary metric $ds^2_{\mathscr{I}} = -dt^2 + d\Omega^2_{S^{d-1}}$
- Null geodesics get to infinity in finite time
- AdS is **not globally hyperbolic**
- Asymptotically AdS spacetimes by definition have the same conformal boundary as AdS



Brief history of AdS

- **AdS metric:** A. Friedmann, *On the possibility of a world with a constant negative curvature of space*, Zeitschrift für Physik 21, 326 (1924)
- *“de Sitter space with negative K involves ideas of altogether too revolutionary a character for physics as it exists today.”*
  J.L. Synge in Relativity: The General Theory (1960)
- Proof of **linear stability** : P. Breitenlohner and D.Z. Freedman, *Stability of gauge extended supergravity*, Annals of Physics 14, 249 (1982)
- Proof of **local well-posedness of the initial-boundary value problem** for 4D vacuum Einstein’s equations with AdS asymptotics:
  H. Friedrich, *Einstein equations and conformal structure: existence of anti-de Sitter-type space-times,* J. Geom. Phys. 17, 125 (1995)
- **AdS/CFT duality**: J. Maldacena, *The large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. 2, 231 (1998)
  (cited 11235 times)

Is AdS stable?

- By the positive energy theorem AdS space is the unique ground state among asymptotically AdS spacetimes (much as Minkowski space is the unique ground state among asymptotically flat spacetimes).
- Basic question for any equilibrium solution: **do small perturbations of it at $t = 0$ remain small for all future times?**
- Minkowski space is asymptotically stable (Christodoulou-Klainerman ’93)
- Key difference between Minkowski and AdS: **the main mechanism of stability of Minkowski - dissipation of energy by dispersion - is absent in AdS** (for no-flux boundary conditions $\mathscr{I}$ acts as a mirror).
- Stability of AdS has not been explored until ’11; notable exceptions: local well-posedness (Friedrich ’95), boundedness of linearized perturbations (Ishibashi-Wald ’04), rigidity (Anderson ’06).
- The problem seems tractable only in spherical symmetry so one needs to add matter to generate dynamics. Simple choice: a scalar field.

AdS gravity coupled to a spherically symmetric scalar field

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R + \Lambda g_{\alpha\beta} = 8\pi T_{\alpha\beta}, \quad \Lambda = -\frac{d(d-1)}{2L^2}$$

$$T_{\alpha\beta} = \partial_\alpha \phi \, \partial_\beta \phi - \frac{1}{2} \left((\partial \phi)^2 + m^2 \phi^2 \right) g_{\alpha\beta}$$

$$\Box_g \phi - m^2 \phi = 0$$

All fields are assumed to be spherically symmetric. Asymptotics near $\mathscr{I}$:

$$\phi(t,x) \sim c_+(t)(\pi/2 - x)^{\frac{d}{2}+\nu} + c_-(t)(\pi/2 - x)^{\frac{d}{2}-\nu}, \quad \nu^2 = \frac{d^2}{4} + m^2 L^2$$

"Reflective" boundary conditions: Dirichlet ($c_- = 0$) or Robin ($c_+ + bc_- = 0$).

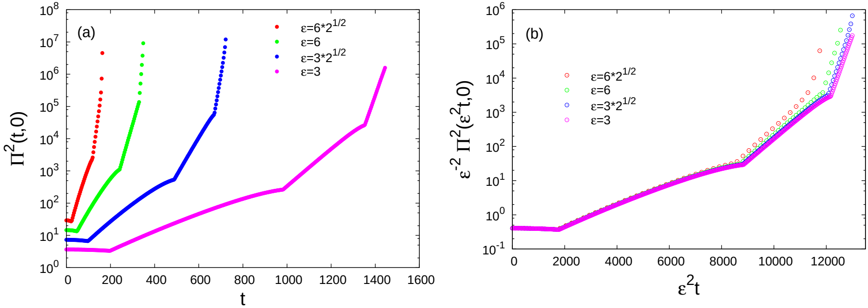
- For $\nu^2 \geq 1$ the initial-boundary value is locally well-posed only for the Dirichlet boundary conditions (Holzegel-Smulevici '11)
- For $\nu^2 = 1/4$ the system is conformally well-behaved at $\mathscr{I}$ and more general boundary conditions (both reflective and dissipative) are allowed (Holzegel-Warnick '13, Holzegel-Luk-Smulevici-Warnick '15).

AdS gravity with a spherically symmetric scalar field

Conjecture (B-Rostworowski ’11)

AdS$_{d+1}$ *(for* $d \geq 3$*) is unstable under arbitrarily small perturbations*

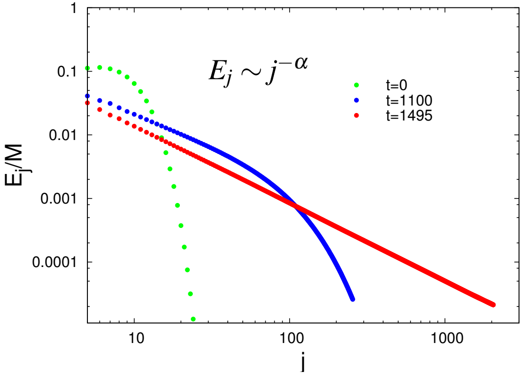
Key numerical evidence:



Gaussian perturbations of size ε collapse in time $\mathscr{O}(\varepsilon^{-2})$.

Heuristic picture

- The linear spectrum is **fully resonant**. Nonlinear interactions between harmonics give rise to **transfer of energy from low to high frequencies**.
- The turbulent cascade leads to concentration of energy on finer and finer spatial scales so eventually a black hole is expected to form.



Evolution of the energy spectrum

Some follow-up studies and open questions

- Turbulent instability is absent for some initial data (**stability islands**). In particular, there exist **stable time-periodic solutions** bifurcating from the eigenmodes (Maliborski-Rostworowski '13).
- Similar phenomenology found for the vacuum Einstein equations in $4+1$ dimensions within the biaxial Bianchi IX ansatz (B-Rostworowski '14).
- What happens outside spherical symmetry? It is not clear at all if the putative endstate of instability - Kerr-AdS black hole - is stable itself. Key issue: **stable trapping** of waves with large angular momentum ℓ:
  - quasinormal modes decay as $e^{-\Gamma_\ell t}$ where $\Gamma_\ell \sim e^{-c\ell}$ (Gannot 2011)
  - linear perturbations decay as $1/\log(t)$ for $t \to \infty$ (Holzegel-Smulevici 2013)
- Is extrapolation of numerical results to *arbitrarily* small perturbations justified? Recently, we (B-Maliborski-Rostworowski '15) validated this extrapolation using so called **resonant approximation**, newly proposed by Balasubramanian et al. and Craps-Evnin-Vanhoof '14. This result hints at a possible route to proving the AdS instability conjecture.

Nonlinear waves in confined geometries

- Consider a nonlinear wave equation for $\phi(t,x)$ with $(t,x) \in R \times M$, where M is a compact Riemannian manifold with metric g.
- Example: $\phi_{tt} - \Delta_g \phi + \phi^3 = 0$ for $M = T^d$ or S^d.
- Goal: understand out-of-equilibrium dynamics of small solutions.
- Due to the lack of dispersion the long-time dynamics is much more complex and mathematically challenging than in the non-compact setting.
- Is the ground state $\phi = 0$ stable (say in H_2 norm)?
- This is an open problem even for $\phi_{tt} - \phi_{xx} + \phi^3 = 0$ on S^1 **!**
- Key enemy: **wave turbulence** - transfer of energy to progressively smaller spatial scales.
- Turbulent solutions: $\limsup_{t\to\infty} \|\phi(t,\cdot)\|_{H^s} = \infty$ for some $s > 1$.

Example: conformal cubic Klein-Gordon on $\mathbb{R} \times \mathbb{S}^3$

$$\partial_{tt}\phi + L\phi + \phi^3 = 0, \qquad L = -\frac{1}{\sin^2 x}\,\partial_x(\sin^2 x\,\partial_x) + 1 \qquad (\star)$$

- Linear spectrum: $Le_n = \omega_n^2 e_n$ where $e_n = \sin(nx)/\sin x$, $\omega_n = n$ ($n \in \mathbb{N}$)
- Plugging the mode expansion $\phi(t,x) = \sum_n c_n(t) e_n(x)$ into ($\star$) we get

$$\frac{d^2 c_n}{dt^2} + \omega_n^2 c_n = \sum_{jkl} I_{jkln}\, c_j c_k c_l, \quad I_{jkln} = -\int_0^\pi e_j(x) e_k(x) e_l(x) e_n(x) \sin^2 x\, dx$$

- In the interaction picture, defined by variation of constants,

$$c_n = \beta_n e^{i\omega_n t} + \bar{\beta}_n e^{-i\omega_n t}, \quad \frac{dc_n}{dt} = i\omega_n \left(\beta_n e^{i\omega_n t} - \bar{\beta}_n e^{-i\omega_n t}\right)$$

  this becomes

$$2i\omega_n \frac{d\beta_n}{dt} = \sum_{jkl} I_{jkln}\, c_j c_k c_l\, e^{-i\omega_n t}$$

- Each term in the sum has a factor $e^{-i\Omega t}$, where $\Omega = \omega_n \pm \omega_j \pm \omega_k \pm \omega_l$. Two kinds of terms: $\Omega = 0$ (**resonant**) and $\Omega \neq 0$ (non-resonant).

Resonant approximation

- We define the slow time $\tau = \varepsilon^2 t$ and rescale $\beta_n(t) = \varepsilon \alpha_n(\tau)$.
- The non-resonant terms $\propto e^{-i\Omega\tau/\varepsilon^2}$ are highly oscillatory for small ε and therefore negligible (at least for some time).
- Keeping only the resonant terms (which is equivalent to time-averaging), we obtain the infinite autonomous dynamical system (**resonant system**)

$$2i\omega_n \frac{d\alpha_n}{d\tau} = \sum_{jkl} I_{jkln}\, \alpha_j \alpha_k \bar{\alpha}_l \,,$$

  where the summation runs over the set of indices $\{jkl\}$ for which $\Omega = 0$ and $I_{jkln} \neq 0$. This set reduces to $\{jkl \,|\, j+k-l=n\}$.
- The resonant system is **invariant under scaling** $\alpha_n(\tau) \to \varepsilon^{-1}\alpha_n(\tau/\varepsilon^2)$
- The resonant approximation is valid on the timescale $\mathcal{O}(\varepsilon^{-2})$. Thus, on this timescale the dynamics of solutions of the conformal cubic Klein-Gordon equation is dominated by resonant interactions.

Resonant approximation for the AdS Einstein-scalar system

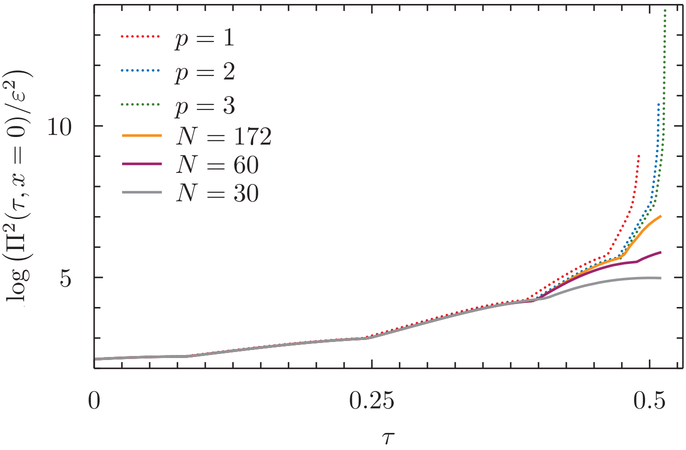
- At the lowest order the resonant system has the same form as above

$$2i\omega_n \frac{d\alpha_n}{d\tau} = \sum_{j+k-l=n} C_{jkln}\, \alpha_j \alpha_k \bar{\alpha}_l\,, \quad (RS)$$

but the interaction coefficients C_{jkln} are *very* complicated (Balasubramanian et al., Craps-Evnin-Vanhoof '14).

- Let $\alpha_n = A_n e^{i\phi_n}$. For large n we have $A_n(\tau) \sim n^{-\alpha(\tau)} e^{-\rho(\tau) n}$, where ρ is the "analyticity radius". If $\lim_{\tau \to \tau_*} \rho(\tau) = 0$ for some τ_* then the solution becomes singular (**analyticity strip method** Sulem-Sulem-Frisch '83).
- Using mixed analytic-numerical methods we showed that for typical initial data $\rho(\tau)$ hits zero in finite time τ_* and $\dfrac{d\phi_n}{d\tau} \sim \ln(\tau_* - \tau)$.
- This indicates that the corresponding solutions of the full system collapse on the timescale $\mathscr{O}(\varepsilon^{-2})$.

How good is the resonant approximation?



Initial data: $\phi(0,x) = \varepsilon\left(\frac{1}{4}e_0(x) + \frac{1}{6}e_1(x)\right)$ with $\varepsilon = 2^{-p}$

Conclusions

- Dynamics of asymptotically AdS spacetimes is an interesting meeting point of fundamental problems in general relativity, PDE theory, and theory of turbulence. Understanding of these connections is at its infancy.
- There is good evidence that the AdS spacetime is unstable against arbitrarily small perturbations (for no-flux boundary conditions at $\mathscr{I}$).
- Understanding of the out-of-equilibrium dynamics of small solutions is mathematically challenging even for the simplest nonlinear wave equations on compact manifolds, let alone Einstein's equations.