From AdS to BEC
(dynamics in spatially confined Hamiltonian systems)

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Unbounded domain	Bounded domain
	
System settles down to equilibrium via dissipation of energy by dispersion	Waves keep interacting for all times, generating out-of-equilibrium dynamics

Understanding of long-time behavior of nonlinear waves in spatially confined systems is challenging. Key questions:

- How the energy injected into the system gets distributed over the degrees of freedom during the evolution?
- Can the energy flow to arbitrarily high frequencies (*weak turbulence*)?

Examples of spatially confined systems

- Nonlinear string

$$\phi_{tt} - \phi_{xx} + \phi^3 = 0, \qquad \phi(t,0) = \phi(t,\pi) = 0$$

- Cubic Klein-Gordon equation on $\mathbb{R} \times S^3$

$$\Box_g \phi - m^2 \phi - \phi^3 = 0, \qquad g = -dt^2 + d\omega^2_{S^3}$$

- Einstein-massless-scalar system with negative cosmological constant

$$R_{\mu\nu} + \lambda g_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi, \qquad \lambda = \frac{d}{l^2}$$

- 2d Gross-Pitaevskii equation with isotropic harmonic potential

$$i\partial_t \Psi = \frac{1}{2}\left(-\partial_x^2 - \partial_y^2 + x^2 + y^2\right)\Psi + g|\Psi|^2\Psi$$

General strategy

- For a spatially confined system, the associated linearized system has a purely discrete spectrum of frequencies
- Expanding solutions in the basis of linear eigenstates one transforms the original PDE into an infinite-dimensional dynamical system with discrete degrees of freedom (‘modes’).
- The nonlinearity generates new frequencies that may lead to resonances between the modes. The resonances dominate the transfer of energy.
- Dropping all nonresonant terms from the Hamiltonian one obtains a simplified infinite-dimensional dynamical system, called the resonant system, which accurately approximates the dynamics of small amplitude solutions of the original PDE on long time scales
- Strategy: try to understand the dynamics of the resonant system and then export this knowledge to the original PDE.

Example

- Background geometry: the Einstein cylinder $\mathscr{M} = \mathbb{R} \times \mathbb{S}^3$ with metric

$$g = -dt^2 + dx^2 + \sin^2 x \, d\omega^2, \qquad (t, x, \omega) \in \mathbb{R} \times [0, \pi] \times \mathbb{S}^2$$

  This spacetime has constant scalar curvature $R(g) = 6$.

- On $\mathscr{M}$ we consider a real scalar field ϕ satisfying

$$\left(\Box_g - \frac{1}{6} R(g)\right) \phi - \phi^3 = \Box_g \phi - \phi - \phi^3 = 0 \, .$$

- We assume that $\phi = \phi(t, x)$. Then, $v(t, x) = \sin(x) \phi(t, x)$ satisfies

$$v_{tt} - v_{xx} + \frac{v^3}{\sin^2 x} = 0$$

  with Dirichlet boundary conditions $v(t, 0) = v(t, \pi) = 0$.

- Linear eigenstates: $e_n(x) = \sqrt{\frac{2}{\pi}} \sin(\omega_n x)$ with $\omega_n = n + 1$ $(n = 0, 1, 2, \ldots)$

Time averaging

- Expanding $v(t,x) = \sum_{n=0}^{\infty} c_n(t) e_n(x)$ we get

$$\frac{d^2 c_n}{dt^2} + \omega_n^2 c_n = -\sum_{jkl} C_{njkl}\, c_j c_k c_l, \quad C_{jkln} = \int_0^{\pi} \frac{dx}{\sin^2 x} e_n(x) e_j(x) e_k(x) e_l(x)$$

- Using variation of constants

$$c_n = \beta_n e^{i\omega_n t} + \bar{\beta}_n e^{-i\omega_n t}, \qquad \frac{dc_n}{dt} = i\omega_n \left(\beta_n e^{i\omega_n t} - \bar{\beta}_n e^{-i\omega_n t}\right)$$

  we factor out fast oscillations

$$2i\omega_n \frac{d\beta_n}{dt} = -\sum_{jkl} C_{njkl}\, c_j c_k c_l\, e^{-i\omega_n t},$$

- Each term in the sum has a factor $e^{-i\Omega t}$, where $\Omega = \omega_n \pm \omega_j \pm \omega_k \pm \omega_l$. The terms with $\Omega = 0$ correspond to resonant interactions.
- Let $\tau = \varepsilon^2 t$ and $\beta_n(t) = \varepsilon \alpha_n(\tau)$. For $\varepsilon \to 0$ the non-resonant terms $\propto e^{-i\Omega\tau/\varepsilon^2}$ are highly oscillatory and therefore negligible.

Resonant system

- Keeping only the resonant terms and rescaling $\alpha_n \to \alpha_n/\sqrt{\omega_n}$, we obtain (B-Craps-Evnin-Hunik-Luyten-Maliborski, 2016)

$$i\,\frac{d\alpha_n}{d\tau} = \sum_{j=0}^{\infty}\sum_{k=0}^{n+j} S_{njk,n+j-k}\,\bar{\alpha}_j\alpha_k\alpha_{n+j-k}\,,$$

  where $S_{njk,n+j-k} = \frac{\min\{n,j,k,n+j-k\}+1}{\sqrt{(n+1)(j+1)(k+1)(n+j-k+1)}}$.

- This system (called the conformal cubic flow) provides an accurate approximation to the cubic wave equation on the timescale $\sim \varepsilon^{-2}$.

- This is a Hamiltonian system

$$i\,\frac{d\alpha_n}{d\tau} = \frac{\partial H}{\partial\bar{\alpha}_n}$$

  with

$$H = \frac{1}{2}\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{n+j} S_{njk,n+j-k}\bar{\alpha}_n\bar{\alpha}_j\alpha_k\alpha_{n+j-k}$$

Other Hamiltonian systems of the form

$$i\,\frac{d\alpha_n}{d\tau} = \sum_{j=0}^{\infty}\sum_{k=0}^{n+j} S_{njk,n+j-k}\,\bar{\alpha}_j\alpha_k\alpha_{n+j-k}$$

- Cubic Szegő equation $S_{njkl} = 1$ (Gérard-Grellier, 2010)
- Lowest Landau Level (LLL) equation: resonant system for the maximally rotating Bose-Einstein condensate (Germain-Hani-Thomann, 2015)

$$S_{njkl} = \frac{(n+j)!}{2^{n+j}\sqrt{n!j!k!l!}}$$

- Resonant system for radial scalar perturbations of AdS_{d+1} spacetime (Balasubramanian et al., Craps-Evnin-Vanhoof, 2014)
- Schrödinger-Newton-Hooke (SNH) system: resonant system for a non-relativistic self-gravitating condensate (B-Evnin-Ficek, 2017)

Basic properties of the systems of the form

$$i\,\frac{d\alpha_n}{d\tau} = \sum_{j=0}^{\infty}\sum_{k=0}^{n+j} S_{njk,n+j-k}\,\bar{\alpha}_j\alpha_k\alpha_{n+j-k}$$

- Symmetries

$$\begin{aligned}\text{Scaling:}\qquad & \alpha_n(t)\to\varepsilon\,\alpha_n(\varepsilon^2 t)\\ \text{Global phase shift:}\qquad & \alpha_n(t)\to e^{i\theta}\alpha_n(t)\\ \text{Local phase shift:}\qquad & \alpha_n(t)\to e^{in\theta}\alpha_n(t)\end{aligned}$$

- Conserved quantities

$$N=\sum_{n=0}^{\infty}|\alpha_n|^2,\qquad Q=\sum_{n=0}^{\infty}(n+1)|\alpha_n|^2$$

- The Szegő, conformal, and LLL flows are locally (and therefore also globally) well-posed for initial data with finite J.
- For Einstein-scalar-AdS resonant system there is evidence that solutions may become singular in finite time.

Finite-dimensional invariant manifolds

- For one-mode initial data $\alpha_n(0) = \delta_{nN}$, the solution is $\alpha_n(\tau) = \delta_{nN} e^{-i\lambda_N \tau}$
- Three-dimensional invariant manifolds: $\alpha_0 = b$ and for $n \geq 1$

$$\alpha_n = \begin{cases} ap^n & \text{Szegő flow} \\ \sqrt{n+1}(bp+an)p^{n-1} & \text{cubic conformal flow} \\ \frac{1}{\sqrt{n!}}(bp+an)p^{n-1} & \text{LLL flow} \end{cases}$$

where the functions $a(\tau), b(\tau), p(\tau)$ are complex-valued.

- The dynamics of these invariant manifolds is described by the reduced Hamiltonian systems

$$\frac{da}{d\tau} = f_1(a,b,p), \quad \frac{db}{d\tau} = f_2(a,b,p), \quad \frac{dp}{d\tau} = f_3(a,b,p)$$

- Since there are three conserved quanities N, Q, and H (that are in involution), the reduced systems are completely integrable.

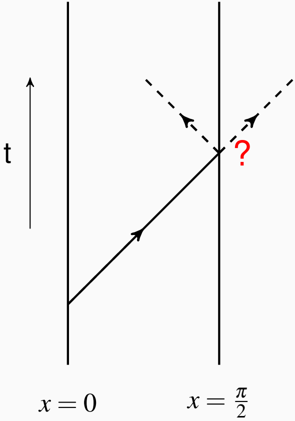
Anti-de Sitter spacetime in $d+1$ dimensions

Manifold $\mathscr{M} = \{t \in \mathbb{R}, x \in [0, \pi/2), \omega \in S^{d-1}\}$ with metric

$$g = \frac{l^2}{\cos^2 x}\left(-dt^2 + dx^2 + \sin^2 x\, d\omega^2_{S^{d-1}}\right)$$

Solution of the vacuum Einstein equations $R_{\alpha\beta} = \lambda g_{\alpha\beta}$ with $\lambda = -d/l^2$.

- Spatial infinity $x = \pi/2$ is the timelike cylinder $\mathscr{I} = \mathbb{R} \times S^{d-1}$ with the boundary metric $ds^2_{\mathscr{I}} = -dt^2 + d\Omega^2_{S^{d-1}}$
- Null geodesics get to infinity in finite time
- Asymptotically AdS spacetimes by definition have the same conformal boundary as AdS
- AdS space is the unique ground state among asymptotically AdS spacetimes.



Conjecture (B-Rostworowski 2011)

AdS_{d+1}, as the solution of the Einstein-massless-scalar field equations with negative cosmological constant in $d+1$ dimensions (for $d \geq 3$), is unstable under arbitrarily small generic perturbations.

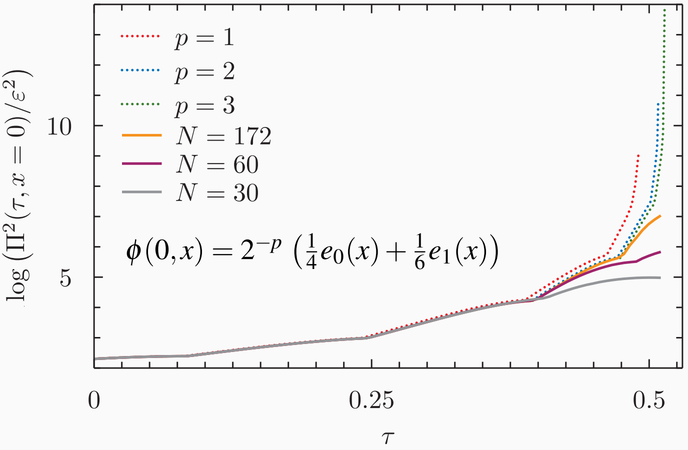
Arguments:

- The linear spectrum is fully resonant. Nonlinear interactions between harmonics give rise to transfer of energy from low to high frequencies.
- The turbulent cascade leads to concentration of energy on finer and finer spatial scales so eventually a black hole is expected to form.
- Numerical evidence: perturbations of size ε collapse in time $\mathcal{O}(\varepsilon^{-2})$.

The shadow of a doubt: is extrapolation to $\varepsilon \to 0$ justified?

Resonant approximation (B-Maliborski-Rostworowski, 2015):

- Using mixed numerical/analytical methods we constructed solutions of the resonant system that become singular in finite time.



- Instability on timescale $1/\varepsilon^2$ is captured by the resonant approximation!
- On the other hand, resonances appear to play no role in the recent proof of instability of AdS for the Einstein-null dust system (Moschidis, 2017)

From Klein-Gordon on AdS to Gross-Pitaevskii

- AdS metric ($r = l \tan x$)

$$g = -(1+\frac{r^2}{l^2})\,dt^2 + \frac{dr^2}{1+\frac{r^2}{l^2}} + r^2 d\omega^2_{\mathbb{S}^{d-1}}$$

- Cubic Klein-Gordon equation on AdS

$$\Box_g \phi - \frac{m^2c^2}{\hbar^2}\,\phi - g|\phi|^2\phi = 0$$

- Substituting $\phi = e^{-i\frac{mc^2}{\hbar}t}\Psi + c.c.$ and taking the limits $l \to \infty$ and $c \to \infty$ such that $c/l \to \omega$, one gets the Gross-Pitaevskii equation

$$i\hbar\partial_t\Psi = -\frac{\hbar^2}{2m}\Delta\Psi + \frac{1}{2}m\omega^2 r^2\Psi + \frac{g\hbar^2}{2m}\,|\Psi|^2\Psi$$

(O. Evnin and G.W. Gibbons, private communication)

Lowest Landau Level equation

- 2d Gross-Pitaevskii equation with isotropic harmonic potential

$$i\partial_t \Psi = \frac{1}{2}\left(-\partial_x^2 - \partial_y^2 + x^2 + y^2\right)\Psi + g|\Psi|^2\Psi$$

- General solution of the linear problem ($g = 0$)

$$\Psi(t,r,\phi) = \sum_{nm} \alpha_{nm}\, e^{-iE_n t} e^{im\phi} \chi_{nm}(r)$$

  where $e^{im\phi}\chi_{nm}(r)$ are normalized eigenstates of energy $E_n = n+1$ and angular momentum $m \in \{-n, -n+2, ..., n-2, n\}$.

- The lowest Landau level (LLL) consists of modes with $m = n$

$$\chi_n(z) = \frac{z^n}{\sqrt{\pi n!}} e^{-\frac{1}{2}|z|^2}, \qquad z = x + iy$$

- The general LLL wavefunction in the frame rotating with angular velocity 1 (where centrifugal and harmonic forces are balanced) is (here $\tau = gt$)

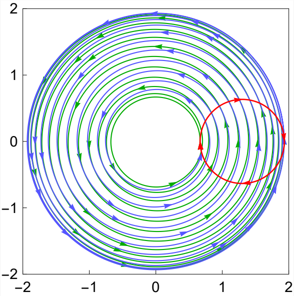
$$\psi(\tau, z) := e^{it}\Psi(t, e^{it}z) = \sum_{n=0}^{\infty} \alpha_n(\tau)\chi_n(z),$$

Vortices in BEC

- A remarkable feature of BEC is the nucleation of quantized vortices when the condensate is stirred above a certain critical angular velocity
- The 3-dimensional invariant manifold of the LLL flow corresponds to single-vortex configurations

$$\psi(\tau, z) = (b(\tau) + a(\tau) z)\, e^{p(\tau) z}\, e^{-\frac{1}{2}|z|^2}$$

- The generic explicit solution represents periodically modulated precession of the vortex
- Motions of this type have been seen in experiments
- It would be very interesting to extend this approach to multi-vortex configurations



Biasi-B-Craps-Evnin, 2017

Schrödinger-Newton-Hooke system

- Einstein-Klein-Gordon system in $3+1$ dimensions with negative cosmological constant $\Lambda = -3/l^2$

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}, \qquad \left(\Box_g - \frac{m^2c^2}{\hbar^2}\right)\phi = 0,$$

  where

$$T_{\alpha\beta} = \partial_\alpha \phi\, \partial_\beta \phi - \frac{1}{2}\left(g^{\mu\nu}\partial_\mu \phi\, \partial_\nu \phi + \frac{m^2c^2}{\hbar^2}\phi^2\right) g_{\alpha\beta}$$

- Substituting $\phi = e^{-i\frac{mc^2}{\hbar}t}\psi + c.c.$ and taking the limits $c \to \infty$ and $l \to \infty$ so that $c/l \to \omega$, we get the SNH system

$$i\hbar\, \partial_t \psi = -\frac{\hbar^2}{2m}\Delta\psi + \frac{1}{2}m\omega^2|x|^2\psi + V\psi, \quad \Delta V = 4\pi G m^2 |\psi|^2$$

  which is equivalent to the Hartree equation with the external harmonic potential (below we set $\hbar = m = G = 1$)

$$i\partial_t \psi = -\frac{1}{2}\Delta\psi + \frac{1}{2}\omega^2|x|^2\psi - \left(|x|^{-1} * |\psi|^2\right)\psi$$

- In higher dimensions $d \geq 3$ the SNH equation reads

$$i\partial_t \psi = -\frac{1}{2}\Delta\psi + \frac{1}{2}\omega^2 |x|^2 \psi - \left(|x|^{-(d-2)} * |\psi|^2\right)\psi$$

- Under scaling $\psi(t,x) \mapsto \psi_\lambda(t,x) = \lambda^{-2}\psi(t/\lambda^2, x/\lambda)$

$$\|\psi_\lambda\|_{L^2} = \lambda^{\frac{d-4}{2}} \|\psi\|_{L^2} \quad \text{and} \quad \|\nabla\psi_\lambda\|_{L^2} = \lambda^{\frac{d-6}{2}} \|\nabla\psi\|_{L^2}$$

  hence the system is L_2-critical for $d = 4$ and energy critical for $d = 6$

- For $d = 4$ the associated resonant system has a three-dimensional invariant manifold on which the dynamics is completely integrable

- In supercritical dimensions $d \geq 7$ we expect a weakly turbulent instability of the zero solution (analogue of AdS instability)

- This expectation is consonant with the ultraviolet asymptotics of the interaction coefficients $S_{nnnn} \sim n^{d-6}$ for $n \to \infty$

Conclusions

- Dynamics of spatially confined Hamiltonian systems is an interesting meeting point of fundamental problems in PDEs and various areas of nonlinear physics
- Despite recent progress, this research area remains largely unexplored

Thank you for your attention