

## Resonant Dynamics and the Instability of Anti-de Sitter Spacetime

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(Received 15 June 2015; published 21 August 2015)

We consider spherically symmetric Einstein-massless-scalar field equations with a negative cosmological constant in five dimensions and analyze the evolution of small perturbations of anti-de Sitter (AdS) spacetime using the recently proposed resonant approximation. We show that for typical initial data the solution of the resonant system develops an oscillatory singularity in finite time. This result hints at a possible route to establishing the instability of AdS under arbitrarily small perturbations.

DOI: 10.1103/PhysRevLett.115.081103

PACS numbers: 04.25.dc, 04.20.Ex

*Introduction.*—A few years ago two of us gave numerical evidence that anti-de Sitter (AdS) spacetime in four dimensions is unstable against black-hole formation for a large class of arbitrarily small perturbations [1]. More precisely, we showed that for a perturbation with amplitude  $\varepsilon$  a black hole forms on the time scale  $\mathcal{O}(\varepsilon^{-2})$ . Using nonlinear perturbation analysis we conjectured that the instability is due to the turbulent cascade of energy from low to high frequencies. This conjecture was extended to higher dimensions in [2].

Because the computational cost of numerical simulations rapidly increases with decreasing  $\varepsilon$ , our conjecture was based on extrapolation of the observed scaling behavior of solutions for small (but not excessively so) amplitudes, which left some room for doubt on whether the instability would persist to arbitrarily small values of  $\varepsilon$  (see, e.g., [3]). To resolve these doubts, in this Letter we validate and reinforce the above extrapolation with the help of a recently proposed resonant approximation [4–6]. The key feature of this approximation is that the underlying infinite dynamical system (hereafter referred to as the resonant system) is scale invariant: if its solution with amplitude 1 does something at time  $t$ , then the corresponding solution with amplitude  $\varepsilon$  does the same thing at time  $t/\varepsilon^2$ . Moreover, the latter solution remains close to the true solution (starting with the same initial data) for times  $\lesssim \varepsilon^{-2}$  (provided that the errors due to omission of higher-order terms do not pile up too rapidly). Thus, by solving the resonant system we can probe the regime of arbitrarily small perturbations (whose outcome of evolution is beyond the possibility of numerical verification).

For concreteness, in this Letter we focus our attention on AdS<sub>5</sub> (the most interesting case from the viewpoint of AdS/CFT correspondence); an extension to other dimensions is straightforward and will be presented elsewhere.

*Model.*—For the reader's convenience, let us recall from [1,2] the general framework for studying the spherically symmetric scalar perturbations of the AdS spacetime. The five-dimensional asymptotically AdS spacetimes are

parametrized by the coordinates  $(t, x, \omega) \in (-\infty, \infty) \times [0, \pi/2) \times \mathbb{S}^3$  and the metric

$$ds^2 = \frac{\ell^2}{\cos^2 x} (-Ae^{-2\delta} dt^2 + A^{-1} dx^2 + \sin^2 x d\omega^2), \quad (1)$$

where  $\ell^2 = -6/\Lambda$ ,  $d\omega^2$  is the round metric on  $\mathbb{S}^3$  and  $A, \delta$  are functions of  $(t, x)$ . For this ansatz the evolution of a self-gravitating massless scalar field  $\phi(t, x)$  is governed by the following system (using units in which  $8\pi G = 3$ ):

$$\partial_t \Phi = \partial_x (Ae^{-\delta} \Pi), \quad \partial_t \Pi = \frac{1}{\tan^3 x} \partial_x (\tan^3 x A e^{-\delta} \Phi), \quad (2)$$

$$\partial_x A = \frac{2 + 2\sin^2 x}{\sin x \cos x} (1 - A) - \sin x \cos x A (\Phi^2 + \Pi^2), \quad (3)$$

$$\partial_x \delta = -\sin x \cos x (\Phi^2 + \Pi^2), \quad (4)$$

where  $\Phi = \partial_x \phi$  and  $\Pi = A^{-1} e^{\delta} \partial_t \phi$ . To ensure smoothness at spatial infinity and finiteness of the total mass  $M$  we impose the boundary conditions (using  $\rho = \pi/2 - x$ )

$$\begin{aligned} \phi(t, x) &= f_\infty(t) \rho^4 + \mathcal{O}(\rho^6), & \delta(t, x) &= \delta_\infty(t) + \mathcal{O}(\rho^8), \\ A(t, x) &= 1 - M\rho^4 + \mathcal{O}(\rho^6), \end{aligned} \quad (5)$$

where the power series expansions are uniquely determined by  $M$  and the functions  $f_\infty(t), \delta_\infty(t)$ . We use the normalization  $\delta(t, 0) = 0$ ; hence,  $t$  is the proper time at the center. We will solve this system for small smooth perturbations of the AdS solution  $\phi = 0, A = 1, \delta = 0$ .

*Resonant approximation.*—As follows from Eq. (2), linearized perturbations of AdS<sub>5</sub> are governed by the operator  $L = -\tan^{-3} x \partial_x (\tan^3 x \partial_x)$ . This operator is essentially self-adjoint with respect to the inner product  $(f, g) := \int_0^{\pi/2} f(x)g(x) \tan^3 x dx$ . The eigenvalues and orthonormal eigenfunctions of  $L$  are  $\omega_n^2 = (2n + 4)^2$  ( $n = 0, 1, \dots$ ) and

$$e_n(x) = 2\sqrt{\frac{(n+2)(n+3)}{n+1}} \cos^4 x P_n^{(1,2)}(\cos(2x)), \quad (6)$$

where  $P_n^{(1,2)}(x)$  is the Jacobi polynomial of order  $n$ .

After these preliminaries we are prepared to introduce the resonant approximation. To avoid technicalities, let us first illustrate this approach in the case of the cubic wave equation on the *fixed* AdS<sub>5</sub> background [7],

$$\partial_t \phi + L\phi + \sec^2 x \phi^3 = 0. \quad (7)$$

Inserting the mode expansion  $\phi(t, x) = \sum_n c_n(t) e_n(x)$  into (7), we get an infinite system of coupled oscillators

$$\frac{d^2 c_n}{dt^2} + \omega_n^2 c_n = \sum_{jkl} I_{jkl n} c_j c_k c_l, \quad (8)$$

where the coefficients  $I_{jkl n} = -(e_j e_k e_l \sec^2 x, e_n)$  determine interactions between the modes. To factor out fast linear oscillations in (8), we change variables using the variation of constants (also known as the “interaction picture”)

$$c_n = \beta_n e^{i\omega_n t} + \bar{\beta}_n e^{-i\omega_n t}, \quad (9)$$

$$\frac{d\beta_n}{dt} = i\omega_n (\beta_n e^{i\omega_n t} - \bar{\beta}_n e^{-i\omega_n t}). \quad (10)$$

This transforms the system (8) into

$$2i\omega_n \frac{d\beta_n}{dt} = \sum_{jkl} I_{jkl n} c_j c_k c_l e^{-i\omega_n t}, \quad (11)$$

where each  $c_j$  in the sum is given by (9); thus, each term in the sum has a factor  $e^{-i\Omega t}$ , where  $\Omega = \omega_n \pm \omega_j \pm \omega_k \pm \omega_l$ . The terms with  $\Omega = 0$  correspond to resonant interactions, while those with  $\Omega \neq 0$  are nonresonant.

Passing to slow time  $\tau = \varepsilon^2 t$  and rescaling  $\beta_n(t) = \varepsilon \alpha_n(\tau)$ , we see that for  $\varepsilon$  going to zero the nonresonant terms  $\propto e^{-i\Omega\tau/\varepsilon^2}$  are highly oscillatory and therefore negligible (at least for some time). Keeping only the resonant terms in (11), which is equivalent to time-averaging, we obtain the infinite autonomous dynamical system (which we shall call the resonant system)

$$2i\omega_n \frac{d\alpha_n}{d\tau} = \sum_{jkl} I_{jkl n} \alpha_j \alpha_k \bar{\alpha}_l, \quad (12)$$

where the summation runs over the set of indices  $\{j, k, l\}$  for which  $\Omega = 0$  and  $I_{jkl n} \neq 0$  (because of the fully resonant nondispersive spectrum of  $L$  and the vanishing of some coefficients  $I_{jkl n}$ , this set reduces to  $\{jkl | j+k-l=n\}$ ; see footnote 3 in [6]). Note that the system (12) is invariant under the scaling  $\alpha_n(\tau) \rightarrow \varepsilon^{-1} \alpha_n(\tau/\varepsilon^2)$ . It is routine to show that

the solutions of (11), starting from small initial data of size  $\varepsilon$ , are well approximated by the solutions of (12) on the time scale  $\mathcal{O}(\varepsilon^{-2})$  [10]. In other words, on this time scale the dynamics of solutions of the cubic wave equation (7) is dominated by resonant interactions.

For the system (2)–(4), the derivation of the resonant system is similar but technically more intricate because, to begin with, one has to integrate the constraints which (at the lowest order) results in nonlocal cubic nonlinearities in the derivatives of  $\phi$ . Despite these complications, the resonant system has the same form as (12), namely

$$2i\omega_n \frac{d\alpha_n}{d\tau} = \sum_{j+k-l=n} C_{jkl n} \alpha_j \alpha_k \bar{\alpha}_l, \quad (13)$$

except that now the interaction coefficients  $C_{jkl n}$  are given by much more complicated expressions involving integrals of products of eigenfunctions (6) and their derivatives. The system (13) was first derived in [4] and [5] using the multiscale perturbation methods [11] and soon afterwards in [6] using the averaging method. It is simpler than the full system, yet still too difficult to be handled by purely analytic means; hence, in what follows we analyze it using numerical and asymptotic methods.

*Results.*—We solved in parallel the full Einstein equations (2)–(4) and the resonant system (13) for a variety of the same small initial data. To illustrate the results (which we believe are universal), we present them for the two-mode initial data with energy (almost) equally distributed between the modes

$$\phi(0, x) = \varepsilon \left( \frac{1}{4} e_0(x) + \frac{1}{6} e_1(x) \right), \quad \Pi(0, x) = 0. \quad (14)$$

Let us point out that, since small one-mode data and their perturbations enjoy quasiperiodic evolutions [1,4,12], the two-mode data are in a sense minimal in what it takes to trigger the turbulent cascade (cf. [13]).

The numerical simulations of the full Einstein equations (2)–(4) were previously reported in [2] in the case of Gaussian initial data. The evolution of the two-mode data (14) looks similar. For all considered small values of  $\varepsilon$ , we observe formation of an apparent horizon (which is signaled by  $A(t, r)$  dropping below a certain small threshold, say  $2^{-16}$ ) in time  $t_H(\varepsilon) \sim \varepsilon^{-2}$  (see Fig. 1). This scaling suggests that the instability should be seen in the resonant approximation. In what follows, we confirm this expectation and, thereby, give support to the conjecture that the instability is present for arbitrarily small perturbations.

For the numerical computation, the resonant system (13) must be truncated at some (possibly large) index  $N$ . As a compromise between the accuracy [14] and the computational cost, we choose here  $N = 172$ . To solve the truncated resonant system (TRS) numerically we use the sixth-order Gauss-Runge-Kutta method [16]. The initial data for the TRS corresponding to (14) are

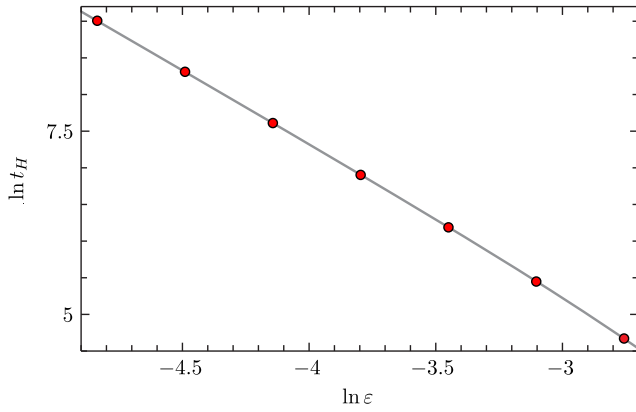


FIG. 1 (color online). Time of horizon formation vs amplitude in the evolution of initial data (14). The solid line depicts the fit of the function  $-2 \ln \epsilon + a + b \epsilon^2$  to the numerical data  $\ln t_H(\epsilon)$ . From this fit we obtain  $\tau_H := \lim_{\epsilon \rightarrow 0} \epsilon^{-2} t_H(\epsilon) = e^a \approx 0.514$ .

$$\alpha_0(0) = 1/8, \quad \alpha_1(0) = 1/12, \quad [\alpha_n(0)]_{n \geq 2} = 0. \quad (15)$$

To describe and analyze the behavior of solutions, it is convenient to use the amplitude-phase representation  $\alpha_n = A_n e^{iB_n}$ , in terms of which the resonant system (13) takes the following form [6]:

$$2\omega_n \frac{dA_n}{d\tau} = \sum_{\substack{j+k-l=n \\ j \neq n, k \neq n}} S_{jkl} A_j A_k A_l \sin(B_n + B_l - B_j - B_k), \quad (16)$$

$$2\omega_n \frac{dB_n}{d\tau} = T_n A_n^2 + \sum_{j \neq n} R_{jn} A_j^2 + A_n^{-1} \sum_{\substack{j+k-l=n \\ j \neq n, k \neq n}} S_{jkl} A_j A_k A_l \cos(B_n + B_l - B_j - B_k), \quad (17)$$

where  $T_n = C_{nnnn}$ ,  $R_{jn} = C_{njjn}$  if  $j \neq n$ , and  $S_{jkl} = C_{jkl}$  if all four indices are different. Explicit expressions for these coefficients are given in Appendix A in [6].

Evolving the initial data (15), we find that higher modes are quickly excited (see Fig. 2). For early times, the amplitudes grow at the polynomial rate  $A_n(\tau) \sim \tau^{n-1}$  while the phases evolve approximately linearly. At later times, the frequencies of oscillations begin to grow rapidly (see Fig. 3). This highly oscillatory behavior accumulates at a finite time, causing numerical difficulties. We find that the time step of numerical integration, for which the algorithm is convergent, tends to zero as the cutoff  $N$  increases. This suggests that the solution of the resonant system ( $N = \infty$ ) develops an oscillatory singularity in some finite time  $\tau_*$  [17].

To give better evidence for the blowup and analyze its character, we proceed in the spirit of the analyticity strip

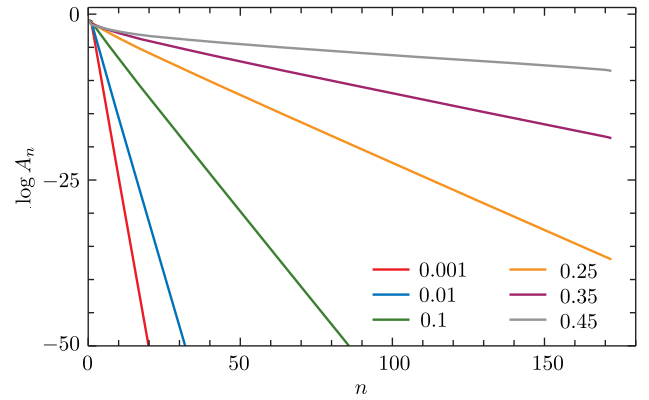


FIG. 2 (color online). The amplitude spectra (for several times  $\tau$ ).

method [18,19]; namely, we make the following asymptotic ansatz for the amplitudes:

$$A_n(\tau) \sim n^{-\gamma(\tau)} e^{-\rho(\tau)n} \quad \text{for } n \gg 1. \quad (18)$$

Fitting this formula to the numerical data, we obtain the time dependence of the exponent  $\gamma(\tau)$  and the ‘‘analyticity radius’’  $\rho(\tau)$ . As shown in Fig. 4, it appears that  $\rho(\tau)$  tends to zero in a finite time  $\tau_*$  (with  $\rho_0 = -\rho'(\tau_*) > 0$ ), confirming that the solution of the resonant system (13) becomes singular at  $\tau_*$ . The fit also reveals that the asymptotic power-law amplitude spectrum has the exponent  $\lim_{\tau \rightarrow \tau_*} \gamma(\tau) = 2$ .

Guided by these numerical findings we will now construct an asymptotic solution of the resonant system that becomes singular in finite time. We assume that for large  $n$  and  $\tau \rightarrow \tau_*$

$$A_n(\tau) \sim n^{-2} e^{-\rho_0(\tau_* - \tau)n}. \quad (19)$$

To solve for the phases, we note the following asymptotic behavior of the interaction coefficients:

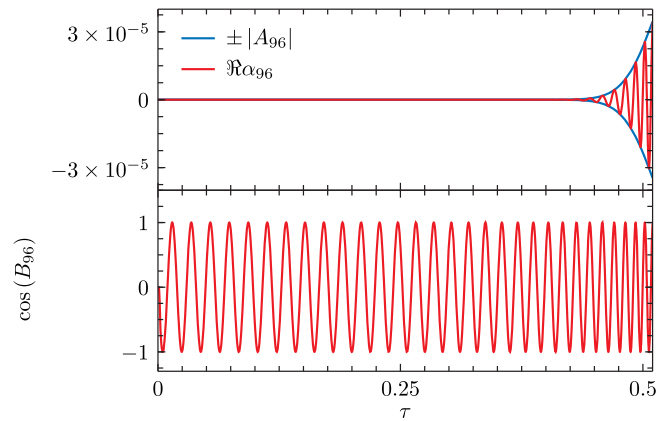


FIG. 3 (color online). Evolution of a sample high mode.

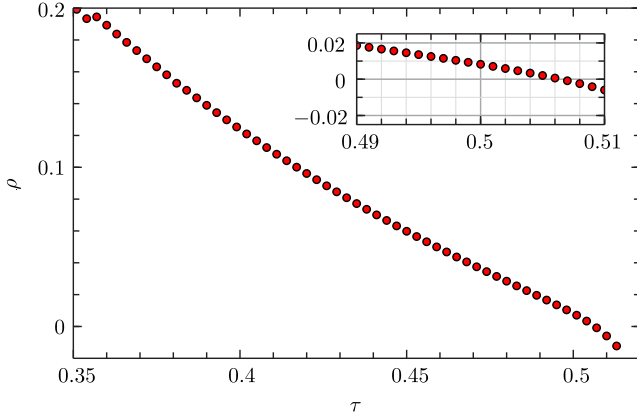


FIG. 4 (color online). The radius of analyticity  $\rho(\tau)$  obtained by fitting the formula (18) to the amplitude spectrum. The point of this (notoriously poor) fit is to show that  $\rho(\tau)$  hits zero in some finite time  $\tau_*$ , not to determine  $\tau_*$  precisely (cf. [17]).

$$T_n \sim n^5, \quad R_{jn} \sim n^2 j^3, \quad S_{\lambda_j, \lambda_k, \lambda_l, \lambda_n} \sim \lambda^4 S_{jkl n}. \quad (20)$$

Notice that the latter implies that

$$\sum_{\substack{j+k=l=n \\ j \neq n, k \neq n}} S_{jkl n} (jkl)^{-2} = \mathcal{O}(1); \quad (21)$$

that is, the sum does not depend on  $n$ . Plugging (19) into Eq. (17) and using (20) and (21), we see that for  $\tau \rightarrow \tau_*$  the rhs of Eq. (17) is dominated by the term

$$\sum R_{jn} A_j^2 \sim n^2 \sum j^{-1} e^{-2\rho_0(\tau_* - \tau)j} \sim n^2 \ln(\tau_* - \tau); \quad (22)$$

thus, the derivatives  $dB_n/d\tau$  blow up logarithmically. Moreover, it follows from the above that for large  $n$  the

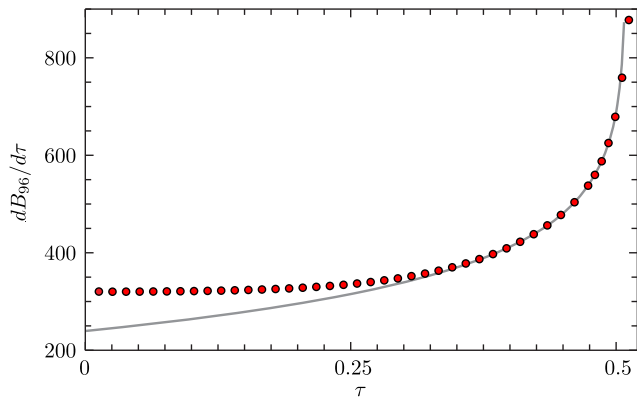


FIG. 5 (color online). Evidence for the logarithmic blowup. The solid line represents the fit of the theoretical prediction  $a_n \ln(\tau_* - \tau) + b_n$  to the numerical data  $dB_n/d\tau$  for  $n = 96$ . Performing this fit for all  $n > 20$  we find, in accordance with the asymptotic analysis, that the coefficients  $a_n$  and  $b_n$  vary linearly with  $n$ , while  $\tau_* \approx 0.509$  does not depend on  $n$ . Note that  $\tau_*$  agrees very well with the time of collapse  $\tau_H$  given in the caption of Fig. 1.

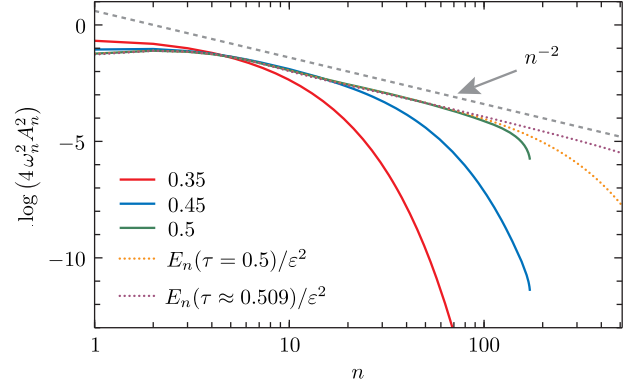


FIG. 6 (color online). Energy spectra for the full system (dotted lines) and TRS (solid lines) at sample late times (cf. Fig. 2). The power-law spectrum  $n^{-2}$  unfolds as the solutions collapse or blow up. Here  $\varepsilon \approx 0.0079$  (the smallest amplitude used in simulations).

phases  $B_n$  behave linearly with  $n$ , hence  $B_n + B_l - B_j - B_k \approx 0$  for the resonant quartets. This implies that both sides of Eq. (16) are (approximately) independent of  $n$ , reassuring us that the ansatz (19) is self-consistent. Numerical simulations indicate that the asymptotics of the blowup just described is in fact universal; this is illustrated in Fig. 5 in the case of two-mode data (15).

*Conclusion.*—To summarize, we have constructed the asymptotic solution of the resonant system that becomes singular in finite time, and we have given numerical evidence that this solution acts as a universal attractor for blowup. The key question is how to transfer this blowup result from the resonant system to the full system. On one hand, we see that the resonant approximation reproduces the amplitudes of true solutions remarkably well almost all the way to collapse. This is illustrated in Figs. 6 and 7, where we compare the energy spectrum and the growth of

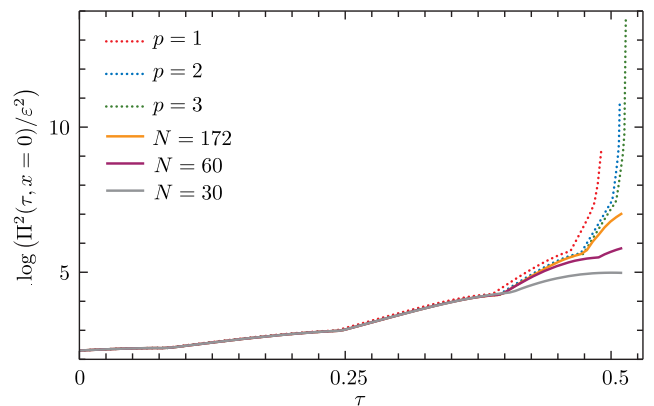


FIG. 7 (color online). Upper envelope of  $\Pi^2(t, 0)$  in the evolution of initial data (14) with  $\varepsilon = (2\pi)^{-3/2} 2^{-p}$  for  $p = 1, 2, 3$  (dotted lines). As  $\varepsilon$  decreases, the rescaled quantities  $\varepsilon^{-2} \Pi^2(\varepsilon^2 t, 0)$  approach a limiting curve. The corresponding solutions of TRS (solid lines) appear to approach the same curve as  $N$  increases.



the Ricci scalar at the origin computed in parallel using the full Einstein equations and the resonant system. On the other hand, the resonant approximation does not work as well for the phases [20]. For this reason (and because of the possible breakdown of the cubic approximation near collapse), it is not clear to us what (if any) is the physical interpretation of the oscillatory singularity for the resonant system [21]. Nonetheless, the fact that solutions of the resonant system blow up in finite time (for typical initial data) strongly indicates that the corresponding solutions of the full system collapse on the time scale  $\mathcal{O}(\varepsilon^{-2})$ .

This work was supported by the Polish National Science Centre Grant No. DEC-2012/06/A/ST2/00397. M. M. acknowledges support from the Alexander von Humboldt Foundation. The computations were carried out with the supercomputer “Zeus” maintained by Academic Computer Centre CYFRONET AGH (Grant No. MNiSW/Zeus\_lokalnie/UJ/027/2014).

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