On blowup of Yang-Mills fields

P. Bizoń

Max-Planck-Institut für Gravitationphysik, Albert-Einstein-Institut, 14476 Golm, Germany
and Institute of Physics, Jagellonian University, Kraków, Poland

Z. Tabor

Department of Biophysics, Jagellonian University, Kraków, Poland

We study the development of singularities for the spherically symmetric Yang-Mills equations in $(d+1)$-dimensional Minkowski spacetime for $d=4$ (the critical dimension) and $d=5$ (the lowest supercritical dimension). Using combined numerical and analytical methods we show in both cases that generic solutions starting with sufficiently large initial data blow up in finite time. The mechanism of singularity formation depends on the dimension: in $d=5$ the blowup is exactly self-similar while in $d=4$ the blowup is only approximately self-similar and can be viewed as the adiabatic shrinking of the marginally stable static solution. The threshold for blowup and the connection with critical phenomena in the gravitational collapse (which motivated this research) are also briefly discussed.

I. INTRODUCTION

The Yang-Mills (YM) equations are the basic equations of gauge theories describing the fundamental forces of nature so understanding their solutions is the issue of great importance. This is not an easy task since, in contrast with Maxwell's equations or the Schrödinger equation, the YM equations are nonlinear, which opens up the possibility that solutions which are initially smooth become singular in the future. Actually such a spontaneous breakdown of the solutions of YM equations cannot occur in the physical $(3+1)$-dimensional Minkowski spacetime as was shown in a classic paper by Eardley and Moncrief [1] who proved that solutions starting from smooth initial data remain smooth for all future times. A natural question is: how the property of global regularity depends on the dimension of the underlying spacetime, in particular, can singularities develop in $d+1$ dimensions for $d>3$? We hope that our paper is a step towards answering this question. As we argue below, the problem of singularity formation for YM equations in higher dimensions is not only interesting in its own right but in addition, it sheds some light onto our understanding of Einstein's equations in the *physical* dimension.

Despite intensive research the problem of global regularity for YM equations in $4+1$ dimensions is entirely open [2]. A lot of progress has been accomplished in proving local existence for "rough" initial data, yet the attempts of proving global regularity for small energy initial data by establishing local well-posedness in the energy norm fail to achieve the goal by "epsilon" [3] (*nota bene* such a local proof of global existence has been obtained in $3+1$ dimensions [4], thereby improving the theorem of Eardley and Moncrief). In this paper we report on numerical simulations which in combination with analytic results strongly suggest that generic solutions with sufficiently large energy do in fact blow up in finite time. Hence, we believe that the smallness condition in the optimal local well-posedness result is not a technical shortcoming but is indispensable. We show that the singularity formation is due to concentration of energy and has the form of adiabatic shrinking of the marginally stable static solution.

Higher dimensions ($d>4$) appear to be somewhat underexplored; the only result we are aware of is the proof of existence of self-similar solutions in $d=5,7,9$ [5]. These solutions provide examples of singularities developing from smooth initial data, however nothing was known about their genericity and stability. Here we restrict ourselves to the $d=5$ case because of its connection with Einstein's equations. We first show that the example of self-similar blowup given in [5] is in fact generic. Then we look at the threshold for singularity formation and observe a behavior similar to the critical behavior in gravitational collapse [6] (with blowup being the analog of a black hole), in particular we find a self-similar solution with one instability as the critical solution.

We remark in passing that there are close parallels between YM equations in $d+1$ dimensions and wave maps in $(d-2)+1$ dimensions [5]. Indeed, many of the phenomena described below have been previously observed by us for the equivariant wave maps into spheres in two [7] and three [8] spatial dimensions.

II. SETUP

We consider Yang-Mills fields in $(d+1)$-dimensional Minkowski spacetime (in the following Latin and Greek indices take the values $1,2,\ldots,d$ and $0,1,2,\ldots,d$ respectively). The gauge potential A_α is a one form with values in the Lie algebra g of a compact Lie group G. Here we take $G=\mathrm{SO}(d)$ so the elements of $g=\mathrm{so}(d)$ can be considered as skew-symmetric $d\times d$ matrices and the Lie bracket is the usual commutator. In terms of the curvature $F_{\alpha\beta}=\partial_\alpha A_\beta - \partial_\beta A_\alpha + e[A_\alpha,A_\beta]$ the Yang-Mills equations are

$$\partial_\alpha F^{\alpha\beta} + e[A_\alpha, F^{\alpha\beta}] = 0, \tag{1}$$

where e is the gauge coupling constant. It is customary to set $e=1$ and we shall also do so in the following. However, it is worth remembering that $[e^2]=M^{-1}L^{d-4}$ (in $c=1$ units); in particular, e^2 has the same dimension in $d=5$ as Newton's constant G in $d=3$.

The YM equations (1) are scale invariant: if $A_\alpha(x)$ is a solution, so is $\tilde{A}_\alpha(x)=\lambda^{-1}A_\alpha(x/\lambda)$. The conserved energy

$$\mathcal{E}(A)=\int_{R^d}\mathrm{Tr}(F_{0i}^2+F_{ij}^2)d^dx \tag{2}$$

scales as $\mathcal{E}(\tilde{A})=\lambda^{d-4}\mathcal{E}(A)$, thus in the partial differential equations (PDE) terminology the YM equations are subcritical for $d\le 3$, critical for $d=4$, and supercritical for $d\ge 5$. It is believed that subcritical equations are globally regular because energy conservation rules out concentration of solutions on arbitrarily small scales. On the contrary, for supercritical equations concentration might be energetically favorable and consequently singularities are expected to occur. The critical equations provide an interesting borderline case.

We assume the spherically symmetric ansatz [9]

$$A_\alpha^{ij}(x)=(\delta_\alpha^i x^j-\delta_\alpha^j x^i)\frac{1-w(t,r)}{r^2}, \tag{3}$$

where $r=\sqrt{x_i^2}$. Then, the YM equations (1) reduce to the scalar semilinear wave equation for the magnetic gauge potential $w(t,r)$,

$$w_{tt}=\Delta_{(d-2)}w+\frac{d-2}{r^2}w(1-w^2)=0, \tag{4}$$

where $\Delta_{(d-2)}=\partial_r^2+[(d-3)/r]\partial_r$ is the radial Laplacian in $d-2$ dimensions. We solve numerically the initial value problem for the above equation in $d=4$ and 5. Our simulations were performed using finite-difference methods combined with adaptive mesh refinement. The latter was essential in resolving the structure of singularities developing on very small scales. To ensure regularity at the center we require that $w(t,0)=1+O(r^2)$. At the outer boundary of the computational grid we impose the outgoing wave condition. Below we present results for the time-symmetric Gaussian initial data of the form

$$w(0,r)=1-Ar^2\exp[-\sigma(r-R)^2], \quad w_t(0,r)=0, \tag{5}$$

with adjustable amplitude A and fixed parameters $\sigma=10$ and $R=2$. We have obtained the same qualitative results for several other families of initial data so we believe that the phenomena described here are generic.

III. RESULTS

We begin our description in a unified dimension-independent manner; all statements which do not explicitly involve the dimension apply both to $d=4$ and $d=5$. Since our data are time symmetric, the initial profile splits into ingoing and outgoing waves. The evolution of the outgoing wave has nothing to do with singularity formation so we shall ignore it in what follows. The behavior of the ingoing wave depends on the amplitude A. For small amplitudes the ingoing wave approaches the center, reaches a minimal radius, bounces back and then disperses to infinity leaving behind an empty space. For large amplitudes the ingoing wave keeps concentrating near the center and eventually blows up in finite time. As the blowup time T is approached we observe the development of a rapidly evolving inner region which is clearly separated from an almost frozen outer region. The inner solution attains a kinklike shape which shrinks in a self-similar manner

$$w(t,r)\approx W(\eta), \quad \eta=\frac{r}{\lambda(t)}, \tag{6}$$

where the profile W depends on the dimension but otherwise seems universal. The scale $\lambda(t)$ goes to zero as $t\to T$ which signals blowup since the second derivative $\partial_r^2 w(t,0)=W''(0)\lambda^{-2}(t)$ becomes unbounded (there is no blowup of the first derivative because $W'(0)=0$, as we shall see below).

The subsequent discussion of the details of blowup has to be given separately in each dimension. We begin with the easier supercritical case.

$d=5$

In this case the scale changes linearly, that is $\lambda(t)=T-t$, as one would expect from dimensional analysis. The blowup profile is given by the exact self-similar solution of Eq. (4)

$$W=W_0(\eta)=\frac{1-\eta^2}{1+\frac{3}{5}\eta^2}, \quad \eta=\frac{r}{T-t}. \tag{7}$$

The solution W_0 was proved to exist in [5] and recently found in closed form by one of us [10]. This solution is
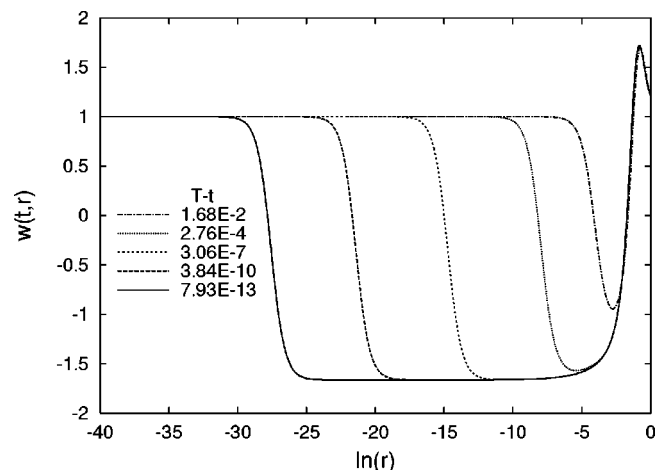

FIG. 1. Blowup in $d=5$. The plot shows the late time evolution of large initial data ($A=0.2$). As the blowup progresses, the inner solution gradually attains the form of the stable self-similar solution $W_0(r/(T-t))$. The outer solution appears frozen on this timescale.
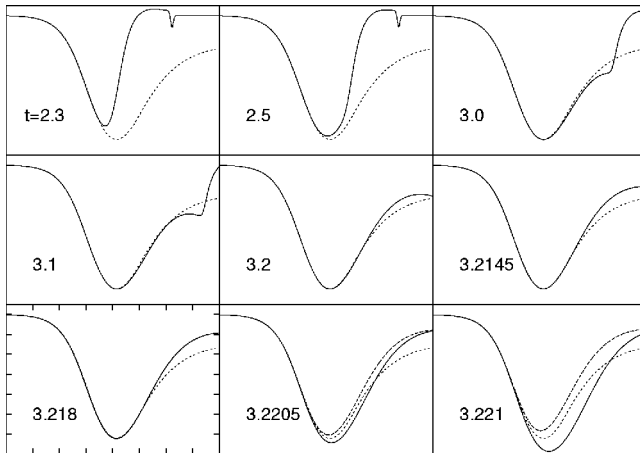
FIG. 2. The critical behavior in $d=5$. The rescaled solution $w(t,(T-t)r)$ is plotted against $\ln(r)$ for a sequence of intermediate times. Shown (solid and dashed lines) is the pair of solutions starting with marginally critical amplitudes $A=A^*\pm\epsilon$, where $A^* =0.144296087005405$. Since $\epsilon=10^{-15}$, the two solutions are indistinguishable on the first seven frames. The convergence to the self-similar solution W_1 (dotted line) is clearly seen in the intermediate asymptotics. The last two frames show the solutions departing from the intermediate attractor towards blowup and dispersion, respectively.

linearly stable [10] (apart from the instability corresponding to shifting the blowup time) which supports the fact that we see it as a generic attractor without tuning any parameters of initial data (see Fig. 1).

We think that the basic mechanism which is responsible for the observed asymptotic self-similarity of blowup can be viewed as the convergence to the lowest "energy" configuration. To see this, note that rewriting Eq. (4) in terms of the similarity variable η and the slow time $\tau=-\ln(T-t)$ one can convert the problem of blowup into the problem of asymptotic behavior of solutions for $\tau\rightarrow\infty$. The point is that in these variables the wave equation contains a damping term which simply reflects the presence of an outward flux of energy through the past light cone of the singularity. Hence it is natural to expect that solutions will tend asymptotically to the least "energy" equilibrium state, which is nothing else but W_0.

It was shown in [10] that W_0 is actually the ground state of a countable family of self-similar solutions W_n ($n =0,1,\dots$) of Eq. (4). All $n>0$ solutions are unstable and therefore not observed in the evolution of generic initial data. However, they may show up in the evolution of specially prepared initial data. The solution W_1 with one unstable mode is particularly interesting since it appears as a transient metastable state in the evolution of initial data tuned to the threshold for blowup. This indicates that W_1 is a critical solution whose codimension-one stable manifold separates blowup from dispersion (see Fig. 2).

Let A^* be the critical amplitude corresponding to the threshold. For initial data with amplitudes near A^*, in the intermediate asymptotics where the solution is approximated by W_1, the amplitude of the unstable mode about W_1 is proportional to $(A^*-A)(T-t)^{-\gamma}$ where $\gamma=5$ (!) is the ei-
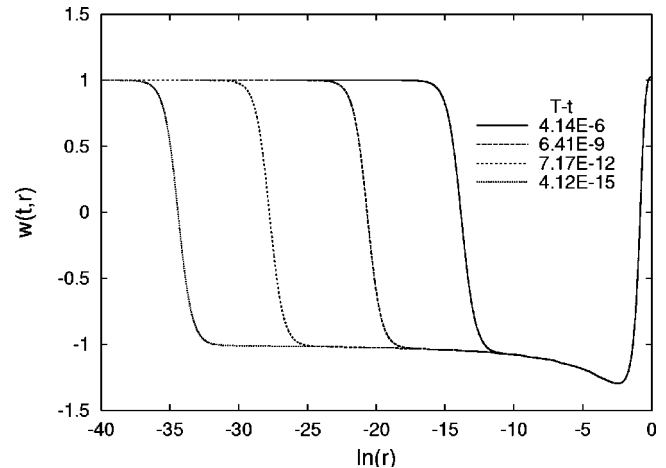

FIG. 3. Blowup in $d=4$. The plot shows the late time evolution of large initial data ($A=0.5$). The inner solution has the form of the scale-evolving static solution $W_S(r/\lambda(t))$ with the scale $\lambda(t)$ going to zero slightly faster than linearly.

genvalue of the unstable mode [10]. This implies that the time of departure from the intermediate attractor, call it t^*, scales as $T-t^*\sim|A^*-A|^{1/5}$. Various scaling laws can be derived from this. For example, consider solutions with marginally subcritical amplitudes $A=A^*-\epsilon$. For such solutions the energy density

$$\rho(t,r)=\frac{w_t^2}{r^2}+\frac{w_r^2}{r^2}+\frac{3(1-w^2)^2}{2r^4} \tag{8}$$

initially grows at the center, attains a maximum at a certain time t_{max} and then drops to zero. An elementary dimensional analysis based on the above scaling predicts the power law $\rho(t_{max},0)\sim\epsilon^{-4/5}$. We have verified this prediction in our simulations (with 4% error).

We point out that the threshold behavior described above shares many features with critical phenomena at the threshold for black hole formation in the gravitational collapse [6]. This fact, together with similar results for wave maps in 3 +1 dimensions [8,11] (and other systems [13]), shows that the basic properties of critical collapse, such as universality, scaling, and self-similarity, originally observed for Einstein's equations, actually have nothing to do with gravity and seem to be robust properties of supercritical evolutionary PDEs.

$d=4$

In this case Eq. (4) does not admit regular self-similar solutions, so the numerically observed self-similarity can be only approximate. We identify the blowup profile as the scale-evolving *static* solution (see Fig. 3)

$$W=W_S(\eta)=\frac{1-\eta^2}{1+\eta^2}, \quad \eta=\frac{r}{\lambda(t)}. \tag{9}$$

We call this solution static because for any *fixed* λ it is the time-independent solution of Eq. (4) (this solution is perhaps better known as the YM instanton in four Euclidean dimen-

sions). Since the energy does not depend on λ, these solutions are only marginally stable; when kicked they shrink or expand. In other words the blowup can be viewed as the adiabatic shrinking of the static solution. Numerical evidence suggests that the rate of blowup goes asymptotically to zero, that is ($\dot{} = d/dt$)

$$\lim_{t \to T} \frac{\lambda(t)}{T-t} = -\lim_{t \to T} \dot{\lambda} = 0. \tag{10}$$

Although we are not able to explain this fact, we point out that it seems necessary for the consistency of the quasi-static character of blowup (9). To see this, substitute the ansatz (6) into Eq. (4) to obtain

$$(1 - \dot{\lambda}^2 \eta^2) W'' + [1 + (\lambda \ddot{\lambda} - 2\dot{\lambda}^2) \eta^2] \frac{W'}{\eta}$$

$$+ \frac{2}{\eta^2} W(1 - W^2) = 0. \tag{11}$$

It follows from Eq. (10) that the terms involving time derivatives of λ in Eq. (11) become asymptotically negligible and therefore in the leading order this equation has the same form as the right-hand side of Eq. (4), which explains why the blowup profile has the shape of the static solution.

In view of the limited resolution of our numerics and the lack of theory we are not in position to make conjectures about the time dependence of λ which would go beyond Eq. (10). Although a power law fit $\lambda \sim (T-t)^{1+\alpha}$ with the anomalous exponent $\alpha \approx 0.1$ is quite accurate, we would not take this fact too seriously because we cannot rule out logarithmic corrections and, moreover, the exponent α exhibits weak dependence on initial data.

It is instructive to compare the $d=4$ and $d=5$ blowups from the standpoint of energy concentration. To this end, for solutions which blow up we define the energy at time $t < T$ inside the past light cone of the singularity

$$\mathcal{E}(t) = c(d) \int_0^{T-t} \left(w_t^2 + w_r^2 + \frac{d-2}{2r^2} (1 - w^2)^2 \right) r^{d-3} dr, \tag{12}$$

where the coefficient $c(d) = (d-1) vol(S^{d-1})$ follows from integrating Eq. (2) over the angles and taking the trace. For $d=5$, substituting Eq. (7) into Eq. (12) we obtain

$\lim_{t \to T} \mathcal{E}(t) = 0$, hence no energy gets concentrated into the singularity. In contrast, for $d=4$, substituting Eq. (9) into Eq. (12) and using Eq. (10) we obtain

$$\lim_{t \to T} \mathcal{E}(t) = 6\pi^2 \int_0^\infty \left(W_S'^2 + \frac{(1 - W_S^2)^2}{r^2} \right) r\, dr = 16\pi^2,$$

thus the energy equal to the energy of the static solution W_S concentrates at the singularity. Note that only the potential energy becomes concentrated; the kinetic energy tends asymptotically to zero. In fact, in our simulations we can see the excess energy being slowly radiated away from the inner region as the blowup profile converges to the static solution.

To summarize, our work provides numerical evidence that solutions of YM equations in four and five spatial dimensions do form singularities from generic smooth large initial data. While the self-similar character of blowup in $d=5$ is well understood (at least from the numerical perspective), the $d=4$ case is more subtle and our analysis leaves two important questions open, namely: what is the precise rate of blowup and what is the nature of the threshold for blowup? We plan to approach these issues by interpreting the expression (9) in terms of motion along the one-dimensional moduli space of static solutions with the scale $\lambda(t)$ playing the role of the collective coordinate. The most straightforward way of computing the dynamics on the moduli space using the geodesic approximation is too naive in the present case: it predicts that λ changes linearly with time [12] which is in contradiction with our numerics. We hope that more refined methods of dealing with collective coordinates, like the ones described in [14] in the context of the nonlinear Schrödinger equation in two spatial dimensions, can be applied to our problem as well. In our opinion, the derivation of the correct modulation equation for the scale λ is the most important next step towards understanding the dynamics of blowup for YM equations in four spatial dimensions; hopefully it would also shed light on the character of transition between blowup and dispersion.

[1] D. Eardley and V. Moncrief, Commun. Math. Phys. **83**, 171 (1982).

[2] S. Klainerman, in *Progress in Nonlinear Differential Equations and Their Applications*, Vol. 29 (Birkhäuser, Basel, 1997).

[3] S. Klainerman and D. Tataru, J. Am. Math. Soc. **12**, 93 (1999).

[4] S. Klainerman and M. Machedon, Ann. Math. **142**, 39 (1995).

[5] T. Cazenave, J. Shatah, and A. Shadi Tahvildar-Zadeh, Ann. Inst. Henri Poincaré Phys. Theor. **68**, 315 (1998).

[6] C. Gundlach, Adv. Theor. Math. Phys. **2**, 1 (1998).

[7] P. Bizoń, T. Chmaj, and Z. Tabor, Nonlinearity **14**, 1041

(2001).

[8] P. Bizoń, T. Chmaj, and Z. Tabor, Nonlinearity **13**, 1411 (2000).

[9] O. Dumitrascu, Stud. Cerc. Mat. **34**(4), 329 (1982).

[10] P. Bizoń (in preparation).

[11] S.L. Liebling, E. Hirschmann, and J. Isenberg, J. Math. Phys. **41**, 5691 (2000).

[12] J.M. Linhart, math-ph/9909015.

[13] M.P. Brenner *et al.*, Nonlinearity **12**, 1071 (1999).

[14] G. Fibich and G. Papanicolaou, SIAM (Soc. Ind. Appl. Math.) J. Appl. Math. **60**, 183 (2000).