

## Note on late-time tails of spherical nonlinear waves

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We consider the longtime behavior of small amplitude solutions of the semilinear wave equation  $\square\phi = \phi^p$  in odd  $d \geq 5$  spatial dimensions. We show that for the quadratic nonlinearity ( $p = 2$ ) the tail has an anomalously small amplitude and fast decay. The extension of the results to more general nonlinearities involving first derivatives is also discussed.

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In a recent paper [1] we studied the late-time tails of spherical waves propagating on even-dimensional Minkowski spacetime under the influence of a long-range radial potential. Using perturbation methods we showed that in six and higher even dimensions there exist exceptional potentials which produce tails with anomalously small amplitudes and fast decay rates. The main purpose of this paper is to show that anomalous tails exist also for nonlinear waves.

We consider the semilinear wave equation with the power nonlinearity

$$\square\phi = \phi^p, \quad \square = \partial_t^2 - \Delta, \quad (1)$$

in odd  $d \geq 5$  spatial dimensions ( $p \geq 2$  is an integer). We assume that initial data are small, smooth, spherically symmetric, and compactly supported

$$\phi(0, r) = \varepsilon f(r), \quad \partial_t \phi(0, r) = \varepsilon g(r). \quad (2)$$

It is well known that the corresponding solutions exist globally in time (see, for instance, [2]) so there arises a natural question: what is the asymptotic behavior of solutions for  $t \rightarrow \infty$ ? In the following we address this question using perturbation theory. Our starting point is the perturbation expansion

$$\phi = \varepsilon\phi_0 + \varepsilon^2\phi_1 + \varepsilon^3\phi_2 + \dots, \quad (3)$$

where  $\varepsilon\phi_0$  satisfies initial data (2) and all higher  $\phi_n$  have zero initial data. Substituting this expansion into Eq. (1) we get the iterative scheme

$$\square\phi_0 = 0, \quad (4a)$$

$$\square\phi_{p-1} = \phi_0^p, \quad (4b)$$

$$\square\phi_{2p-2} = p\phi_0^{p-1}\phi_{p-1}, \quad \text{etc.}, \quad (4c)$$

which can be solved recursively.

The general spherically symmetric solution of Eq. (4a) is given by the superposition of outgoing and ingoing waves

$$\phi_0(t, r) = \phi_0^{\text{ret}}(t, r) + \phi_0^{\text{adv}}(t, r), \quad (5)$$

where

$$\begin{aligned} \phi_0^{\text{ret}}(t, r) &= \frac{1}{r^{l+1}} \sum_{k=0}^l \frac{(2l-k)!}{k!(l-k)!} \frac{a^{(k)}(u)}{(v-u)^{l-k}}, \\ \phi_0^{\text{adv}}(t, r) &= \frac{1}{r^{l+1}} \sum_{k=0}^l (-1)^{k+1} \frac{(2l-k)!}{k!(l-k)!} \frac{a^{(k)}(v)}{(v-u)^{l-k}}, \end{aligned} \quad (6)$$

and  $u = t - r$ ,  $v = t + r$  are the retarded and advanced times, respectively (the superscript in round brackets denotes the  $k$ -th derivative). Here and in the following, it is convenient to use the positive integer index  $l$  defined by  $d = 2l + 3$  (recall that we consider only odd spatial dimensions  $d \geq 5$ ). Note that for compactly supported initial data the generating function  $a(x)$  can be chosen to have compact support as well (this condition determines  $a(x)$  uniquely).

To solve Eq. (4b) we use the Duhamel formula for the solution of the inhomogeneous equation  $\square\phi = N(t, r)$  with zero initial data (see, e.g., [3])

$$\phi(t, r) = \frac{1}{2r^{l+1}} \int_0^t d\tau \int_{|t-r-\tau|}^{t+r-\tau} \rho^{l+1} P_l(\mu) N(\tau, \rho) d\rho, \quad (7)$$

where  $P_l(\mu)$  are Legendre polynomials of degree  $l$  and  $\mu = (r^2 + \rho^2 - (t - \tau)^2)/2r\rho$  (note that  $-1 \leq \mu \leq 1$  within the integration range). Applying this formula to Eq. (4b) and using null coordinates  $\eta = \tau - \rho$  and  $\xi = \tau + \rho$  we obtain

$$\begin{aligned} \phi_{p-1}(t, r) &= \frac{1}{2^{l+3}r^{l+1}} \int_{|t-r|}^{t+r} d\xi \int_{-\xi}^{t-r} (\xi - \eta)^{l+1} P_l(\mu) \\ &\quad \times \phi_0^p(\eta, \xi) d\eta, \end{aligned} \quad (8)$$

where now  $\mu = (r^2 + (\xi - t)(t - \eta))/r(\xi - \eta)$ . If the initial data (2) vanish outside a ball of radius  $R$ , then for  $t > r + R$  we may drop the advanced part of  $\phi_0$  and interchange the order of integration in (8) to get

$$\phi_{p-1}(t, r) = \frac{1}{2^{l+3} r^{l+1}} \int_{-\infty}^{\infty} d\eta \int_{t-r}^{t+r} (\xi - \eta)^{l+1} P_l(\mu) [\phi_0^{\text{ret}}(\eta, \xi)]^p d\xi. \quad (9)$$

Substituting (5) into (9) and using the identity (see the Appendix for the derivation),<sup>1</sup>

$$\begin{aligned} \int_{t-r}^{t+r} d\xi \frac{P_l(\mu)}{(\xi - \eta)^n} &= (-1)^l \frac{2(n-2)^l}{(2l+1)!!} r^{l+1} \frac{(t-\eta)^{n-l-2}}{[(t-\eta)^2 - r^2]^{n-1}} F\left(\frac{l+2-n}{2}, \frac{l+3-n}{2} \middle| \left(\frac{r}{t-\eta}\right)^2\right) \\ &= (-1)^l \frac{2(n-2)^l}{(2l+1)!!} \frac{r^{l+1}}{t^{l+n}} \left(1 + (l+n) \frac{\eta}{t} + \mathcal{O}\left(\frac{1}{t^2}\right)\right), \end{aligned} \quad (10)$$

we get

$$\phi_{p-1}(t, r) = \frac{C(l, p)}{t^{(l+1)p-1}} \left[ I_l(p, 0) + \mathcal{O}\left(\frac{1}{t}\right) \right], \quad (11)$$

where

$$C(l, p) := (-1)^l \frac{2^{(l+1)(p-1)-1}}{(2l+1)!!} [(l+1)(p-1) - 2]^l, \quad (12)$$

and

$$I_l(p, q) := \int_{-\infty}^{+\infty} (a^{(l)}(\eta))^p (a^{(l+1)}(\eta))^q d\eta. \quad (13)$$

The coefficient  $I_l(p, q)$  is the only trace of initial data. We point out that there is no loss of generality in putting the coefficient  $C(l, p)$  outside the square bracket in (11) because if  $C(l, p) = 0$  (which happens for  $p = 2$ ), then the integrand over  $\eta$  in (9) becomes a total derivative, hence the whole integral (9) vanishes for compactly supported initial data.

It is not difficult to verify that generically  $\phi_{2p-2}$  and all higher-order iterates also decay as  $1/t^{(l+1)p-1}$ , thus  $\phi_{p-1}$  gives a good approximation of the full tail provided that  $\varepsilon$  is sufficiently small. More precisely, for fixed  $r$  and  $t \rightarrow \infty$  we have

$$\phi(t, r) \approx \varepsilon^p \phi_{p-1}(t, r), \quad (14)$$

up to an error of order  $\mathcal{O}(t^{-(l+1)p-1})\mathcal{O}(\varepsilon^{2p-1})$ . We remark that for  $l = 0$  the series (3) was proven in [4] to converge for small enough  $\varepsilon$ , thereby making the approximation (14) rigorous. We have not been able to prove an analogous convergence result for  $l \geq 1$ ; however, from the practical point of view the asymptotic nature of the perturbation series is sufficient in using the approximation (14) to make quantitative physical predictions.

The advantage of the approach presented above, in contrast to decay estimates in the form of inequalities, is that it makes it easy to identify anomalous tails for which the amplitude of the leading-order term in the perturbation expansion vanishes. In the case at hand, as mentioned above, this happens for the quadratic nonlinearity  $p = 2$  since from (12) the coefficient  $C(l, 2)$  vanishes for any

<sup>1</sup>As in [1] we use the notation  $x^0 := 1$ ,  $x^k := x \cdot (x-1) \cdot \dots \cdot (x-(k-1))$ ,  $k > 0$ .

$l \geq 1$ . This implies that there is no tail in the first order or, put differently, the system of Eqs. (4a) and (4b) satisfies Huygens' principle (note that this is not true in three spatial dimensions, i.e., for  $l = 0$ , [4]).

In order to determine the tail for the quadratic nonlinearity we need to go to the second order. Applying the Duhamel formula (7) to Eq. (4c) we obtain

$$\begin{aligned} \phi_2(t, r) &= \frac{1}{2^{l+2} r^{l+1}} \int_{|t-r|}^{t+r} d\xi \int_{-\xi}^{t-r} (\xi - \eta)^{l+1} P_l(\mu) \\ &\quad \times \phi_0(\eta, \xi) \phi_1(\eta, \xi) d\eta. \end{aligned} \quad (15)$$

To compute the asymptotic behavior of this expression near timelike infinity we need to find first the asymptotic expansion of  $\phi_1(t, r)$  near null infinity ( $u = \text{const}$  and  $v \rightarrow \infty$ ). Substituting (6) into (4b) we get

$$\phi_1(u, v) = \text{free part} + \frac{h(u)}{(v-u)^{2l+1}} + \mathcal{O}(1/v^{2l+2}), \quad (16)$$

$$h(u) = -\frac{2^{2l}}{l} \int_{-\infty}^u [a^{(l)}(x)]^2 dx.$$

Plugging (6) and (16) into (15), interchanging the order of integration, and expanding in powers of  $1/t$ , we get the leading-order asymptotic behavior at timelike infinity

$$\phi_2(t, r) = (-1)^l \frac{2^{3l}}{2l(2l+1)} \frac{1}{t^{3l+1}} \left[ I_{l-1}(1, 2) + \mathcal{O}\left(\frac{1}{t}\right) \right]. \quad (17)$$

Thus, for  $p = 2$  the approximation (14) should be replaced by

$$\phi(t, r) \approx \varepsilon^3 \phi_2(t, r) + \mathcal{O}(\varepsilon^4), \quad (18)$$

where  $\phi_2$  is given by (17).

Since quadratic nonlinearities are common in nonlinear perturbation analysis, the anomalous tail (18) appears frequently in applications. We emphasize that these applications are not restricted to higher dimensions and might be physically relevant because some important equations in physics (we mean, in four dimensions) are equivalent to scalar wave equations in higher dimensions (actually, we first came across this phenomenon while studying the Yang-Mills equations in four dimensions [5]).

The analysis presented above can be readily generalized to incorporate nonlinearities involving derivatives, for ex-

ample, equations of the form

$$\square\phi = \phi^p(\alpha\partial_t\phi + \beta\partial_r\phi)^q, \quad (19)$$

where integers  $p$  and  $q$  satisfy  $p + q \geq 2$ , and  $\alpha, \beta$  are constants. For this equation, proceeding along the same lines as in the derivation of the tail (11), we obtain

$$\phi_{p+q-1}(t, r) = (\alpha - \beta)^q \frac{C(l, p + q)}{t^{(l+1)(p+q)-1}} \left[ I_l(p, q) + \mathcal{O}\left(\frac{1}{t}\right) \right]. \quad (20)$$

There are several special cases when the tail (20) vanishes and the decay is faster:

- (i)  $q = 0$  and  $p = 2$ . This case has been discussed above (see (17)).
- (ii)  $q = 1$  and  $p \geq 1$ . In this case the coefficient  $I_l(p, 1)$  vanishes (since the integrand in (13) is the total derivative and  $a(x)$  is compactly supported) and, instead of (20), we have

$$\phi_p(t, r) = \frac{D(l, p)}{t^{(l+1)(p+1)}} \left[ I_l(p + 1, 0) + \mathcal{O}\left(\frac{1}{t}\right) \right], \quad (21)$$

where

$$D(l, p) = (-1)^l \frac{2^{(l+1)p-1}}{(2l+1)!!} (l+1)[(l+1)p-1]^l \times \left[ (\beta - \alpha) \frac{p-1}{p+1} \frac{(l+1)(p+1)-1}{(l+1)p-1} - 2\beta \right]. \quad (22)$$

Note that for  $p = 1$  and  $\beta = 0$  the coefficient  $D(l, p)$  vanishes. In this case there is no tail at the first order whatsoever, while at the second order, in analogy to (17), we obtain

$$\phi_2(t, r) = (-1)^l \alpha^2 2^{3l-2} \frac{3l+1}{2l(2l+1)} \frac{1}{t^{3l+2}} \times \left[ I_l(3, 0) + \mathcal{O}\left(\frac{1}{t}\right) \right]. \quad (23)$$

- (iii)  $q = 2$  and  $p = 0$ . In this case  $C(l, p + q) = 0$  and, instead of (20), we have

$$\phi_1(t, r) = (-1)^l \alpha \beta \frac{2^{l+2} l!}{(2l+1)!!} (l+1)^3 \frac{1}{t^{2l+3}} \times \left[ I_l(2, 0) + \mathcal{O}\left(\frac{1}{t}\right) \right], \quad (24)$$

thus for  $\alpha\beta = 0$  there is no first-order tail, in analogy to the case  $q = 0$  and  $p = 2$ . At the second order we get

$$\phi_2(t, r) = (-1)^{l+1} (\alpha - \beta)^4 \frac{2^{3l}}{2l(2l+1)} \frac{1}{t^{3l+1}} \times \left[ I_l(0, 3) + \mathcal{O}\left(\frac{1}{t}\right) \right]. \quad (25)$$

We note that for  $l = 1$  and  $0 \neq \alpha \neq \beta \neq 0$  this case is exceptional in the sense that the first-order tail decays faster than the second-order tail. This is, in fact, a peculiar property of the nonlinearity of the form  $\partial_t\phi\partial_r\phi$  in 5 + 1 dimensions, as  $(\alpha\partial_t\phi + \beta\partial_r\phi)^2 = \alpha^2(\partial_t\phi)^2 + 2\alpha\beta\partial_t\phi\partial_r\phi + \beta^2(\partial_r\phi)^2$ . Note also that from (24) and (25) the nonlinearities  $(\partial_t\phi)^2$  and  $(\partial_r\phi)^2$  produce exactly the same tail, in agreement with the well-known fact that the equation  $\square\phi = (\partial_t\phi)^2 - (\partial_r\phi)^2$  is Huygensian.

- (iv)  $q \geq 1$  and  $\alpha = \beta \neq 0$ . In this case we have

$$\phi_{p+q-1}(t, r) = \frac{E(l, p, q)}{t^{(l+1)(p+q)+q-1}} \times \left[ I_l(p + q, 0) + \mathcal{O}\left(\frac{1}{t}\right) \right], \quad (26)$$

where

$$E(l, p, q) = (-1)^{l+q} \alpha^q \frac{2^{(l+1)(p+q-1)+q-1}}{(2l+1)!!} (l+1)^q \times ((l+1)(p+q-1) + q - 2)^l. \quad (27)$$

The formula (26) reduces to (21) if  $q = 1$  and  $p = 1$ , and to (24) if  $q = 2$  and  $p = 0$ . Finally, we wish to remark that all the above analytic predictions have been scrupulously verified numerically.

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## APPENDIX

Here we derive the identity (10). Changing the integration variable from  $\xi$  to  $\mu$  we get

$$\int_{t-r}^{t+r} d\xi \frac{P_l(\mu)}{(\xi - \eta)^n} = \frac{r(t - \eta)^{n-2}}{[(t - \eta)^2 - r^2]^{n-1}} \int_{-1}^{+1} d\mu P_l(\mu) \left(1 - \frac{r\mu}{t - \eta}\right)^{n-2}. \quad (A1)$$

Expanding the power and integrating using the identity [6]

$$\mu^k = \sum_{l=k, k-2, k-4, \dots} \frac{(2l+1)k!}{2^{(k-l)/2} \left(\frac{k-l}{2}\right)! (k+l+1)!} P_l(\mu), \quad (A2)$$

we obtain

$$\int_{t-r}^{t+r} d\xi \frac{P_l(\mu)}{(\xi - \eta)^n} = (-1)^l r^{l+1} \frac{(t - \eta)^{n-l-2}}{[(t - \eta)^2 - r^2]^{n-1}} \sum_{0 \leq m} 2^{1-m} \binom{n-2}{l+2m} \frac{(l+2m)!}{m!(2l+2m+1)!!} \left(\frac{r}{t-\eta}\right)^{2m}. \quad (\text{A3})$$

Expressing the sum in terms of the hypergeometric function we get (10).

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