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AdS collapse of a scalar field in higher dimensions

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We show that the weakly turbulent instability of anti–de Sitter space, recently found in P. Bizoń and A. Rostworowski, Phys. Rev. Lett. **107**, 031102 (2011) for 3 + 1-dimensional spherically symmetric Einstein-massless-scalar field equations with negative cosmological constant, is present in all dimensions $d + 1$ for $d \geq 3$.

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In a recent paper [1] two of us reported on numerical simulations that indicate that anti–de Sitter (AdS) space in 3 + 1 dimensions is unstable against the formation of a black hole under arbitrarily small generic perturbations. This instability was conjectured to be triggered by a resonant mode mixing that moves energy from low to high frequencies. While the case of 3 + 1 dimensions is the most interesting from the point of view of general relativity, it is natural to ask whether a corresponding result holds in higher $d + 1$ dimensions, in particular, in the case $d = 4$ that is the most interesting from the point of view of the AdS/CFT correspondence. In this paper we answer this question in the positive, that is we show that AdS_{d+1} is unstable under arbitrarily small generic perturbations in all supercritical dimensions $d \geq 3$. To demonstrate that, we first generalize the formalism of [1] to higher dimensions. Second, we recall from [1] the key mechanism that generates instability and point out that this mechanism operates in all dimensions $d \geq 3$. Third, we support this analytic argument by numerical simulations of weakly perturbed AdS_5. Finally, we comment on a recent work by Garfinkle and Pando Zayas [2] who looked at the same problem but did not find instability for small perturbations.

We parametrize the $(d + 1)$-dimensional asymptotically AdS metric by the ansatz

$$ds^2 = \frac{\ell^2}{\cos^2 x}(-Ae^{-2\delta}dt^2 + A^{-1}dx^2 + \sin^2 x\, d\Omega^2_{d-1}), \quad (1)$$

where $\ell^2 = -d(d-1)/(2\Lambda)$, $d\Omega^2_{d-1}$ is the round metric on S^{d-1}, $-\infty < t < \infty$, $0 \leq x < \pi/2$, and A, δ are functions of (t, x). For this ansatz the evolution of a self-gravitating massless scalar field $\phi(t, x)$ is governed by the following system (using units in which $8\pi G = d - 1$):

$$\dot{\Phi} = (Ae^{-\delta}\Pi)', \qquad \dot{\Pi} = \frac{1}{\tan^{d-1}x}(\tan^{d-1}x\, Ae^{-\delta}\Phi)', \quad (2)$$

$$A' = \frac{d-2+2\sin^2 x}{\sin x \cos x}(1-A) - \sin x \cos x\, A(\Phi^2 + \Pi^2), \quad (3)$$

$$\delta' = -\sin x \cos x(\Phi^2 + \Pi^2), \quad (4)$$

where $\dot{} = \partial_t$, $' = \partial_x$, $\Phi = \phi'$, and $\Pi = A^{-1}e^{\delta}\dot{\phi}$. We want to solve the system (2)–(4) for small smooth perturbations of AdS solution $\phi = 0$, $A = 1$, $\delta = 0$. Smoothness at the center implies that near $x = 0$

$$\phi(t, x) = f_0(t) + \mathcal{O}(x^2), \qquad \delta(t, x) = \mathcal{O}(x^2),$$
$$A(t, x) = 1 + \mathcal{O}(x^2), \quad (5)$$

where we used normalization $\delta(t, 0) = 0$ so that t is the proper time at the center. Smoothness at spatial infinity and finiteness of the total mass M imply that near $x = \pi/2$ we must have (using $\rho = \pi/2 - x$)

$$\phi(t, x) = f_\infty(t)\rho^d + \mathcal{O}(\rho^{d+2}), \qquad \delta(t, x) = \delta_\infty(t) + \mathcal{O}(\rho^{2d}),$$
$$A(t, x) = 1 - M\rho^d + \mathcal{O}(\rho^{d+2}), \quad (6)$$

where the power series expansions are uniquely determined by M and the functions $f_\infty(t)$, $\delta_\infty(t)$ (which in turn are determined by the evolution of initial data). One can show that the initial-boundary value problem for the system (2)–(4) together with the regularity conditions (5) and (6) is locally well posed.

In [1] the instability of AdS_4 was conjectured to result from the resonant mode mixing that moves energy from low to high frequencies. It was argued that this process of energy concentration on increasingly small spatial scales must be eventually cut off by the formation of a black hole. It is easy to see that the same mechanism is at work for all $d \geq 3$. This follows from the fact that, using the PDE terminology, the system (2)–(4) is fully resonant. More precisely, the spectrum of the linear self-adjoint operator, which governs the evolution of linearized perturbations of AdS_{d+1}, $L = -\tan^{1-d}x\,\partial_x(\tan^{d-1}x\,\partial_x)$, is given by $\omega_j^2 = (d + 2j)^2$, $(j = 0, 1, \ldots)$. The key point is that the frequencies ω_j are equally spaced, so already at the third order of nonlinear perturbation analysis one gets resonant terms for any frequency ω_j such that $j = j_1 + j_2 - j_3$, where j_k are indices of eigenmodes present in the initial data. Some of these resonances lead to secular terms that signal the onset of instability at time $t = \mathcal{O}(\varepsilon^{-2})$, where ε measures the size of initial data.

To substantiate this heuristic argument we solve the system (2)–(4) in $d = 4$ numerically, using the method of [1]. For easy comparison of our results with those of Garfinkle and Pando Zayas, we take the same approximately ingoing Gaussian initial data as Ref. [2]: $\Phi(0, x) = \partial_x \phi(0, x) = \Pi(0, x)$, where

$$\phi(0, x) = \frac{\varepsilon}{\sqrt{3}} \exp\left(-\frac{(\tan x - r_0)^2}{\sigma^2}\right) \tag{7}$$

with $r_0 = 4$, $\sigma = 1.5$. As in [2] we set the AdS radius $\ell = 1$, hence their and our radial coordinates are related by $r = \tan x$, while the time coordinates are identical. We denote their amplitude A by ε (to avoid conflict of notation); the factor $1/\sqrt{3}$ comes from the difference in units: we use $8\pi G = 3$, while in [2] $8\pi G = 1$. Note that the data (7) slightly violate the regularity condition (5) since $\Phi(0, 0)$ is not exactly zero; however, an error generated by this "corner singularity" is negligible.

For large ε, the solution quickly collapses (the formation of an apparent horizon is detected by the metric function $A(t, x)$ touching zero at some x_H). As ε decreases, the horizon radius takes the form of the right continuous sawtooth curve $x_H(\varepsilon)$ (see Fig. 1 of [1]) with jumps at critical points ε_n where $\lim_{\varepsilon \to \varepsilon_n^+} x_H(\varepsilon) = 0$ (the index n counts the number of reflections from the AdS boundary before collapse). Accordingly, the time of horizon formation $t_H(\varepsilon)$ is a monotone decreasing piecewise continuous function with jumps at each ε_n (see Fig. 1).

For small initial data the weakly nonlinear perturbation analysis described in [1] predicts the onset of instability at time $t \sim \varepsilon^{-2}$. Numerics indicates that for sufficiently small ε this scaling holds approximately almost all the way to collapse, that is $t_H(\varepsilon) \sim \varepsilon^{-2}$ as well. The evidence for this fact is shown in Fig. 2 that depicts the evolution of three solutions with small amplitudes differing by a factor of $\sqrt{2}$.

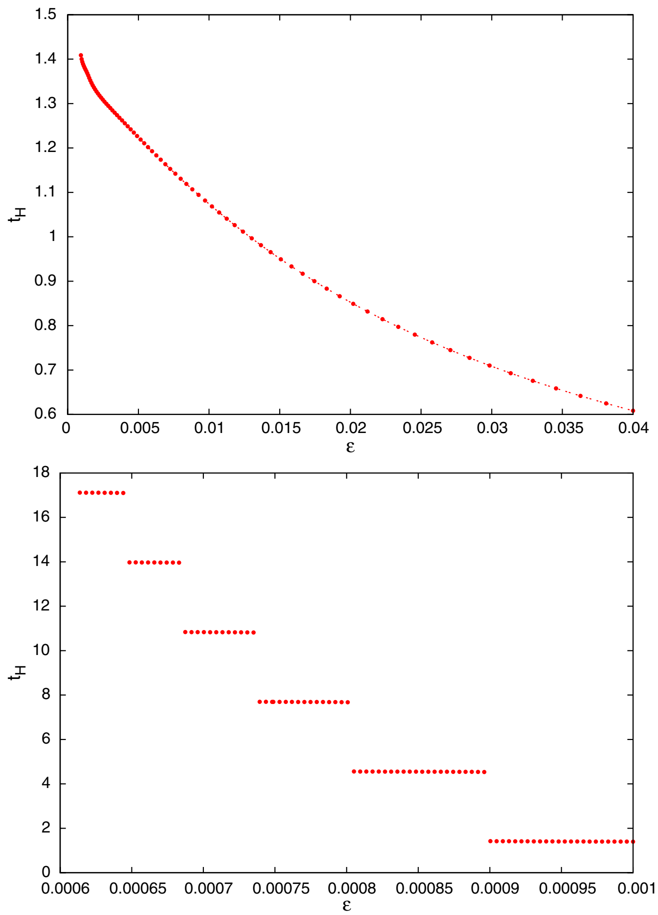

FIG. 1 (color online). Time of horizon formation t_H vs amplitude for initial data (7). (Top) First "step" of the "staircase" function $t_H(\varepsilon)$ corresponding to large data solutions that collapse during the first implosion. The horizon radius varies from $x_H = 0.5$ to zero (from right to left). For $\varepsilon > 0.005$, the plot coincides with Fig. 4 of [2] which verifies that our results agree with those of [2] for short enough times. (Bottom) A few further steps of $t_H(\varepsilon)$ corresponding to solutions that bounce several times from the AdS boundary before collapsing.

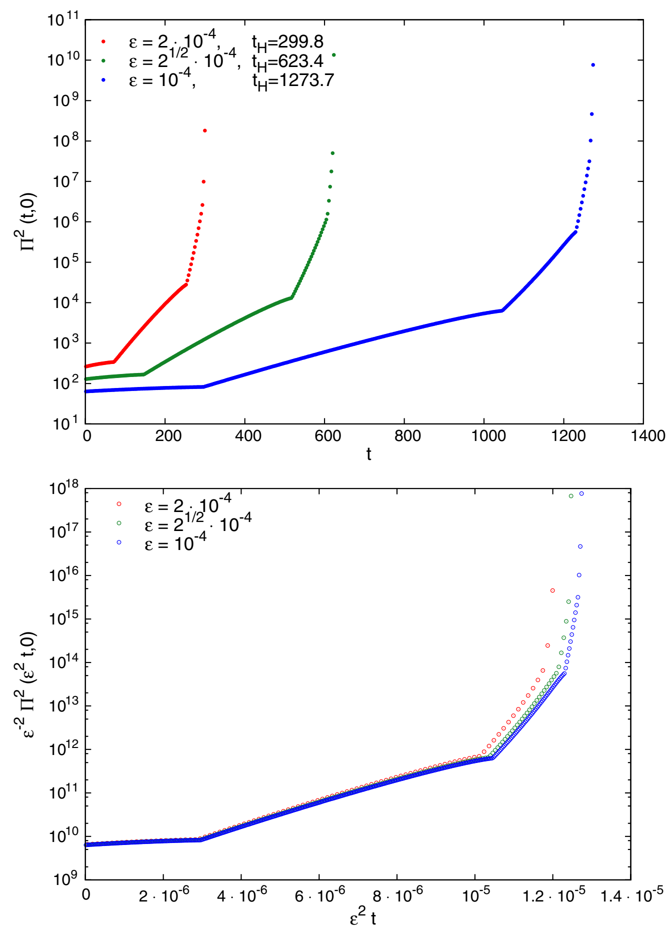

FIG. 2 (color online). (Top) Upper envelope of $\Pi^2(t, 0)$ for initial data (7) with three relatively small amplitudes. After making 95 (for $\varepsilon = 0.0002$), 198 (for $\varepsilon = \sqrt{2} \cdot 0.0001$), and 405 (for $\varepsilon = 0.0001$) reflections, all solutions eventually collapse. (Bottom) Curves from the upper plot after rescaling $\varepsilon^{-2}\Pi^2(\varepsilon^2 t, 0)$ seem to converge to a limiting curve.

Note that this scaling implies that the computational cost of numerical evolution increases rapidly as ε decreases (since solutions have to be evolved for longer and longer times on finer and finer grids).

Finally, let us comment on the paper [2]. The abstract of that paper states: ''We [...] establish that for small values of the initial amplitude of the scalar field there is no [*sic*] black hole formation, rather, the scalar field performs an oscillatory motion typical of geodesics in AdS.'' This assertion, as we have shown above, is false so one might wonder what led the authors of [2] to reach this conclusion. The only ''evidence'' is given in Fig. 2 of [2], which depicts the solution with small amplitude $\varepsilon = 0.0002$ bouncing 4 times without forming a horizon. Concluding from this that ''the scalar field performs an oscillatory motion'' forever and a horizon will never form is a completely unjustified extrapolation. Our numerical evolution of the same data (shown in Fig. 2) yields horizon formation at $x_H \approx 4.8 \times 10^{-4}$ after time $t_H \approx 299.8$. The numerical method used by Garfinkle and Pando Zayas was too crude to simulate evolution for so long times. Their code is based on the second order finite difference method on a fixed nonuniform grid using uncompactified radial coordinate r. The AdS timelike boundary at $r = \infty$ is mimicked by an artificial reflecting mirror at $r_{\max} = 10$, which introduces a small error at each bounce (since the pulse gets reflected before reaching the true boundary). However, the key obstacle that prevented the authors of [2] to run reliable simulations for longer times was the lack of sufficient resolution of their code. In particular, their code could not capture the spatio-temporal structure of horizon formation for the data with amplitude $\varepsilon = 0.0002$ since this structure develops below the first point of the grid. For comparison, our fourth-order code with global adaptive mesh refinement reached the level of $2^{17} + 1$ grid points just before collapse. The lesson is that the long-time numerical simulations of asymptotically AdS spacetimes are challenging even in spherical symmetry, and one should be careful in jumping to conclusions about the late time dynamics, especially without an analytic understanding of the problem.

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[1] P. Bizoń and A. Rostworowski, Phys. Rev. Lett. **107**, 031102 (2011).

[2] D. Garfinkle and L. A. Pando Zayas, Phys. Rev. D **84**, 066006 (2011).