Anomalously small wave tails in higher dimensions

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We consider the late-time tails of spherical waves propagating on even-dimensional Minkowski spacetime under the influence of a long range radial potential. We show that in six and higher even dimensions there exist exceptional potentials for which the tail has an anomalously small amplitude and fast decay. Tails outside higher-dimensional Schwarzschild black holes are also discussed.

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I. INTRODUCTION

It is well-known that sharp propagation of free waves along light cones in even-dimensional flat spacetimes, known as Huygens' property, is blurred by the presence of a potential. Physically, the spreading of waves inside the light cone is caused by the backscattering off the potential. If the potential falls off exponentially or faster at spatial infinity, then the backscattered waves decay exponentially in time, while the long range potentials with an algebraic fall-off give rise to tails which decay polynomially in $1/t$. The precise description of these tails is an important issue in scattering theory. There are two main approaches to this problem in the literature. On the one hand, there are mathematical results in the form of various decay estimates. These results are rigorous, however they rarely give optimal decay rates inside the light cone and provide very poor information about the amplitudes of tails. The notable exception is the work of Strauss and Tsutaya [1] (recently strengthened by Szpak [2]) where the optimal pointwise decay estimate for the tail was proved in four dimensions. Unfortunately, to the best of our knowledge, there is no analogous result in higher dimensions.

On the other hand, there are nonrigorous results in the physics literature based on perturbation theory. The most complete work in this category was done by Ching *et al.* [3] who derived first-order approximations of the tails for radial potentials. Although these results were originally formulated for partial waves in four dimensions, they can be easily translated to spherical waves in higher dimensions. Ching *et al.* noticed that there are exceptional potentials for which the first-order tail vanishes, however they did not pursue their analysis to the second order, apart from giving some dimensional arguments. The main purpose of this paper is to analyze the tails for such exceptional potentials in more detail.

One of the physical motivations behind our work stems from the fact that these kinds of potentials arise in the study of linearized perturbations of higher even-dimensional Schwarzschild black holes. The behavior of tails on the Schwarzschild background is well-known in four dimensions (see [4–8]), but not in higher even dimensions (despite statements to the contrary in the literature [9]). Although our analysis is restricted to the flat background, it sheds some light on the problem of tails on the black hole background because the properties of tails are to some extent independent of what happens in the central region.

The rest of the paper is organized as follows. In Sec. II we construct the iterative scheme for the perturbation expansion of a spherically symmetric solution of the linear wave equation with a potential. This scheme is applied in Sec. III to derive the first- and second-order approximations of the tails for radial potentials which fall off as pure inverse-power at infinity. In Sec. IV we discuss the modifications caused by subleading terms in the potential. Section V contains numerical evidence confirming the analytic formulas from Secs. III and IV. Finally, in Sec. VI we give a heuristic argument to predict the behavior of tails outside Schwarzschild black holes in higher even dimensions. Technical details of most calculations are given in the appendix.

Throughout the paper we use the succinct notation and summation techniques from the excellent book by Graham *et al.* [10]. In particular, we shall frequently use the following abbreviations:

$$x^{\underline{0}} := 1, \qquad x^{\underline{k}} := x(x-1)\cdots(x-(k-1)), \qquad k>0, \tag{1}$$

$$x^{\overline{0}} := 1, \qquad x^{\overline{k}} := x(x+1)\cdots(x+(k-1)), \qquad k>0. \tag{2}$$

II. ITERATIVE SCHEME

We consider the wave equation with a potential in even-dimensional Minkowski spacetime R^{d+1},

$$\partial_t^2\phi - \Delta\phi + \lambda V\phi = 0. \tag{3}$$

The prefactor λ is introduced for convenience—throughout the paper we assume that λ is small which allows us to use it as the perturbation parameter. The precise assumptions about the fall-off of the potential will be formulated below. We restrict attention to spherical symmetry, i.e., we assume that $\phi = \phi(t, r)$ and $V = V(r)$. Then, Eq. (3) becomes

$$\mathcal{L}\,\phi + \lambda V(r)\phi = 0, \qquad \mathcal{L} := \partial_t^2 - \partial_r^2 - \frac{d-1}{r}\partial_r. \tag{4}$$

We are interested in the late-time behavior of $\phi(t, r)$ for smooth compactly supported (or exponentially localized) initial data

$$\phi(0, r) = f(r), \qquad \partial_t\phi(0, r) = g(r). \tag{5}$$

To determine the asymptotic behavior of solutions we define the perturbative expansion (Born series)

$$\phi = \sum_{n=0} \lambda^n \phi_n, \tag{6}$$

where ϕ_0 satisfies initial data (5) and all ϕ_n with $n > 0$ have zero initial data. Substituting this expansion into Eq. (4) we get the iterative scheme

$$\mathcal{L}\,\phi_n = -V\phi_{n-1}, \qquad \phi_{-1} = 0, \tag{7}$$

which can be solved recursively. The zeroth-order solution is given by the general regular solution of the free radial wave equation which is a superposition of outgoing and ingoing waves [11]

$$\phi_0(t, r) = \phi_0^{\mathrm{ret}}(t, r) + \phi_0^{\mathrm{adv}}(t, r), \tag{8}$$

where

$$\begin{aligned} \phi_0^{\mathrm{ret}}(t, r) &= \frac{1}{r^{l+1}} \sum_{k=0}^{l} \frac{(2l-k)!}{k!(l-k)!} \frac{a^{(k)}(u)}{(v-u)^{l-k}}, \\ \phi_0^{\mathrm{adv}}(t, r) &= \frac{1}{r^{l+1}} \sum_{k=0}^{l} (-1)^{k+1} \frac{(2l-k)!}{k!(l-k)!} \frac{a^{(k)}(v)}{(v-u)^{l-k}}, \end{aligned} \tag{9}$$

and $u = t - r$, $v = t + r$ are the retarded and advanced times, respectively. Here and in the following, instead of d, we use the index l defined by $d = 2l + 3$ (remember that we consider only *odd* space dimensions d). Note that for compactly supported initial data the generating function $a(x)$ can be chosen to have compact support as well (this condition determines $a(x)$ uniquely).

To solve Eq. (7) for the higher-order perturbations we use the Duhamel representation for the solution of the inhomogeneous equation $\mathcal{L}\phi = N(t, r)$ with zero initial data,

$$\phi(t, r) = \frac{1}{2r^{l+1}} \int_0^t d\tau \int_{|t-r-\tau|}^{t+r-\tau} \rho^{l+1} P_l(\mu) N(\tau, \rho) d\rho, \tag{10}$$

where $P_l(\mu)$ are Legendre polynomials of degree l and $\mu = (r^2 + \rho^2 - (t-\tau)^2)/2r\rho$ (note that $-1 \le \mu \le 1$ within the integration range). This formula can be readily obtained by integrating out the angular variables in the standard formula $\phi = G^{\mathrm{ret}} * N$, where $G^{\mathrm{ret}}(t, x) = (2\pi^{l+1})^{-1}\Theta(t)\delta^{(l)}(t^2 - |x|^2)$ is the retarded Green's function of the wave operator in $d + 1$ dimensions (see, for example, [12]).

It is convenient to express (10) in terms of null coordinates $\eta = \tau - \rho$ and $\xi = \tau + \rho$:

$$\phi(t, r) = \frac{1}{2^{l+3} r^{l+1}} \int_{|t-r|}^{t+r} d\xi \int_{-\xi}^{t-r} (\xi - \eta)^{l+1} P_l(\mu) \times N(\eta, \xi) d\eta, \tag{11}$$

where now $\mu = (r^2 + (\xi - t)(t - \eta))/r(\xi - \eta)$. Using this representation we can rewrite the iterative scheme (7) in the integral form

$$\phi_n(t, r) = -\frac{1}{2^{l+3} r^{l+1}} \int_{|t-r|}^{t+r} d\xi \int_{-\xi}^{t-r} (\xi - \eta)^{l+1} P_l(\mu) \times V(\rho(\eta, \xi))\phi_{n-1}(\eta, \xi) d\eta. \tag{12}$$

This "master" equation will be applied below to evaluate the first two iterations for a special class of potentials. It is natural to expect that for sufficiently small λ these iterations provide good approximations of the true solution.

III. PURE INVERSE-POWER POTENTIALS AT INFINITY

In this section we consider the simple case (below referred to as type I) when the potential is *exactly* $V(r) = r^{-\alpha}$ for r greater than some $r_0 > 0$. We assume that $\alpha > 2$. The modifications caused by subleading corrections to the pure inverse-power decay of the potential will be discussed in Sec. IV.

A. Generic case

We wish to evaluate the first iteration $\phi_1(t, r)$ near timelike infinity, i.e., for $r = \mathrm{const}$ and $t \to \infty$. Thanks to the fact that $\phi_0(\eta, \xi)$ has compact support we may interchange the order of integration in (12) and drop the advanced part of $\phi_0(\eta, \xi)$ to obtain

$$\phi_1(t, r) = -\frac{2^\alpha}{2^{l+3} r^{l+1}} \int_{-\infty}^{\infty} d\eta \int_{t-r}^{t+r} (\xi - \eta)^{l+1-\alpha} P_l(\mu) \times \phi_0^{\mathrm{ret}}(\eta, \xi) d\xi, \tag{13}$$

where we have substituted $V = 2^\alpha(\xi - \eta)^{-\alpha}$. Plugging (9) into (13), after a long calculation (see the appendix for the technical details), we get

$$\phi_1(t,r) = -2^{\alpha+3l-1}\left(\frac{\alpha-3}{2}\right)^{\underline{l}}\left(\frac{\alpha}{2}\right)^{\overline{l}} \int_{-\infty}^{+\infty} d\eta a(\eta) \frac{(t-\eta)^{\alpha-2}}{[(t-\eta)^2-r^2]^{\alpha-1+l}} \sum_{0\le n\le[(\alpha-2)/2]} (-1)^n \frac{2^{2n}(l+n)!}{n!(2l+2n+1)!}$$
$$\times\left(-\frac{\alpha-2}{2}-l-1\right)^{\underline{n}}\left(\frac{\alpha-1}{2}-l-1\right)^{\underline{n}} \sum_{n\le m\le n+l}(-1)^m\binom{l}{m-n}\frac{(-\frac{\alpha}{2}+1)^{\overline{m}}}{(\frac{\alpha}{2})^{\overline{m}}}\left(\frac{r}{t-\eta}\right)^{2m}. \tag{14}$$

Asymptotic expansion of (14) near timelike infinity yields the following first-order approximation of the tail:

$$\phi(t,r) \approx \lambda\phi_1(t,r) = \lambda\frac{C(l,\alpha)}{t^{\alpha+2l}}\left[A + (\alpha+2l)\frac{B}{t} + \mathcal{O}\left(\frac{1}{t^2}\right)\right], \tag{15}$$

where

$$C(l,\alpha) = -\frac{2^{\alpha+2l-1}}{(2l+1)!!}\left(\frac{\alpha-3}{2}\right)^{\underline{l}}\left(\frac{\alpha}{2}\right)^{\overline{l}}, \tag{16}$$

and

$$A = \int_{-\infty}^{+\infty} a(\eta)d\eta, \qquad B = \int_{-\infty}^{+\infty} a(\eta)\eta d\eta. \tag{17}$$

In general $A \neq 0$ and the tail decays as $t^{-\alpha-2l}$, however there are nongeneric initial data for which $A = 0$ and then the tail decays as $t^{-\alpha-2l-1}$; in particular, this happens for time symmetric initial data for which $a(x)$ is an odd function.

Remark 1. It is easy to check that if the function $\phi(t,r)$ satisfies Eq. (4), then the function $\psi = r^{l+1}\phi$ satisfies the radial wave equation for the lth multipole

$$(\partial_t^2 - \partial_r^2 + l(l+1)/r^2)\psi + \lambda V(r)\psi = 0. \tag{18}$$

The late-time tails for this equation were studied by Ching *et al.* [3] who derived the formula equivalent to (15) via the Fourier transform methods.

B. Exceptional case

It follows from (15) that if α is an odd integer satisfying $3 \le \alpha \le 2l+1$, then $\phi_1(t,r)$ vanishes identically due to the factor $(\frac{\alpha-3}{2})^{\underline{l}}$ in (16) and there is no (polynomial) tail whatsoever in the first order. Thus, in order to compute the tail in this exceptional case we need to go to the second order of the perturbation expansion.

Using (12) and proceeding as above we get the second iteration

$$\phi_2(t,r) = -\frac{2^\alpha}{2^{l+3}r^{l+1}}\int_{-\infty}^{\infty} d\eta \int_{t-r}^{t+r}(\xi-\eta)^{l+1-\alpha}P_l(\mu) \times \phi_1^{\mathrm{ret}}(\eta,\xi)d\xi, \tag{19}$$

where ϕ_1^{ret} is the outgoing solution of the inhomogeneous equation

$$\mathcal{L}\phi_1 = -V\phi_0. \tag{20}$$

In general ϕ_1 is a sum of the solution of the homogeneous equation and the particular solution of the inhomogeneous equation. The homogeneous part has the form (9) (with a different generating function than a, but still compactly supported), thus for the same reason as above it gives no contribution to the tail. The particular solution of the inhomogeneous Eq. (20) reads

$$\phi_l^{NH} = \frac{1}{2(\alpha-1)r^{\alpha+l}}\sum_{q=0}^{l-\alpha/2+1/2}(l-\alpha/2+1/2)^{\underline{q}} \times \frac{2^q(\alpha/2)^{\overline{q}}}{\alpha^{\overline{q}}}\frac{\phi_{l-1-q}^H}{r^q}, \tag{21}$$

where ϕ_{l-1-q}^H denotes the solution of the homogeneous equation with $d = 2(l-1-q)+3$ and the same generating function a as in ϕ_0 [see (9)]. The formula (21) can be easily derived by the method of undetermined coefficients (we emphasize that this formula is valid *only* for odd α satisfying $3 \le \alpha \le 2l+1$). Substituting (21) into (19), after a long calculation (see the appendix for the technical details), we obtain the following asymptotic behavior near timelike infinity:

$$\phi(t,r) \approx \lambda^2\phi_2(t,r) = \lambda^2\frac{D(l,\alpha)}{t^{2(\alpha+l-1)}}\left[A + 2(\alpha+l-1)\frac{B}{t} + \mathcal{O}\left(\frac{1}{t^2}\right)\right], \tag{22}$$

where the coefficients A and B are defined in (17) and (see Table I)

$$D(l,\alpha) = \frac{2^{2(\alpha+l-2)}}{(2l+1)!!}\cdot\frac{(2\alpha-3)}{2(\alpha-1)}\left(\alpha-\frac{5}{2}\right)^{\underline{l-1}}(\alpha-2+l)^{\underline{l-1}} \times F\left(\left.\begin{matrix}-l+\alpha/2-1/2, \alpha/2, 2\alpha-2, 1\\ \alpha, \alpha, \alpha-l-1/2\end{matrix}\right| 1\right). \tag{23}$$

Here F stands for the generalized hypergeometric function

$$F\left(\left.\begin{matrix}a_1,\ldots,a_m\\ b_1,\ldots,b_n\end{matrix}\right| z\right) = \sum_{k\ge 0}\frac{a_1^{\overline{k}},\ldots,a_m^{\overline{k}}}{b_1^{\overline{k}},\ldots,b_n^{\overline{k}}}\frac{z^k}{k!}. \tag{24}$$

TABLE I. The first few coefficients $D(l,\alpha)$.

	$\alpha = 3$	$\alpha = 5$	$\alpha = 7$	$\alpha = 9$
$l = 1$	4			
$l = 2$	$-8/5$	$2240/3$		
$l = 3$	$96/35$	1792	$2\,523\,136/5$	
$l = 4$	$-64/7$	$-17\,920/9$	$16\,580\,608/5$	$4\,638\,965\,760/7$

We remark that the behavior $\mathcal{O}(\lambda^2)t^{-2(l+\alpha-1)}$ of the tail (22) was conjectured before by Ching *et al.* [3] on the basis of dimensional analysis.

IV. GENERAL POLYNOMIALLY DECAYING POTENTIALS

In this section we analyze how the presence of subleading corrections to the pure inverse-power asymptotic behavior of the potential affects the results obtained in Sec. III. We restrict ourselves to the most interesting and common case (below referred to as type II) when near infinity:

$$V(r) = \frac{1}{r^\alpha}\left(1 + \frac{\beta}{r^\gamma}\right) + o\left(\frac{1}{r^{\alpha+\gamma}}\right), \qquad \gamma > 0. \tag{25}$$

If $C(\alpha, l) \neq 0$, then the dominant behavior of the tail is of course the same as in (15):

$$\phi(t, r) \sim \lambda A C(l, \alpha) t^{-(\alpha+2l)}. \tag{26}$$

However, in the exceptional case, when $C(\alpha, l) = 0$, the situation is more delicate. As we showed above, in this case there is the second-order contribution to the tail given by (22),

$$\phi_2(t, r) \sim A D(l, \alpha) t^{-2(\alpha+l-1)}. \tag{27}$$

In contrast to the type I case where the first-order tail vanishes identically, in the type II case the subleading term in the potential produces the first-order contribution which is given by (15) with α replaced by $\alpha + \gamma$:

$$\phi_1(t, r) \sim \beta A C(l, \alpha + \gamma) t^{-(\alpha+\gamma+2l)}, \tag{28}$$

assuming that $\alpha + \gamma$ is not an odd integer $\leq d - 2$ (otherwise one has to repeat the analysis for the next subleading term in the potential).

Now, comparing the decay rates in (27) and (28) we conclude that the leading asymptotics of the tail is given by the first-order term $\lambda\phi_1(t, r)$ if $\gamma \leq \alpha - 2$ (we call it subtype IIa), but otherwise, i.e. for $\gamma > \alpha - 2$ (subtype IIb), the second-order term $\lambda^2\phi_2(t, r)$ is dominant for $t \to \infty$.

Remark 2. In the context of Eq. (18) a formula analogous to (28) was obtained by Hod who studied tails in the presence of subleading terms in the potential (see subgroup IIIb in [13]). However, Hod's analysis, restricted to the first-order approximation, was inconclusive because, as we just have shown, without the second-order formula (27) one is not in position to make assertions about the dominant behavior of the tail.

V. NUMERICS

In order to verify the above analytic predictions we solved numerically the initial value problem (4) and (5) for various potentials and initial data. Our numerical algorithm is based on the method of lines with finite differencing in space and explicit fourth-order accurate Runge-Kutta time stepping. As was pointed out in [3], a reliable numerical computation of tails requires high-order finite-difference schemes, since otherwise the ghost potentials generated by discretization errors produce artificial tails which might mask the genuine behavior. The minimal order of spatial finite-difference operators depends on the fall-off of the potential—for the cases presented below the fourth-order accuracy was sufficient, but for the faster decaying potentials a higher-order accuracy is needed. To eliminate high-frequency numerical instabilities we added a small amount of Kreiss-Oliger artificial dissipation. All computations were performed using quadruple precision which was essential in suppressing round-off errors at late times.

The numerical results presented here were produced for initial data of the form

$$\phi(0, r) = \exp(-r^2), \qquad \partial_t\phi(0, r) = \exp(-r^2). \tag{29}$$

As follows from (9) the generating function for these data is

$$a(x) = 2^{-(l+2)}(1 - 2x)\exp(-x^2), \quad \text{hence } A = \int_{-\infty}^{+\infty} a(x)dx = \sqrt{\pi}/2^{l+2}. \tag{30}$$

We considered the following potentials:

$$V(r) = \frac{\tanh^{\alpha+2} r}{r^\alpha} \qquad \text{(type I)} \tag{31a}$$

$$V(r) = \frac{\tanh^{\alpha+2} r}{r^\alpha}\left(1 + \frac{\tanh^\gamma r}{r^\gamma}\right) \qquad \text{(type II)} \tag{31b}$$

for various values of α and γ. The regularizing factor $\tanh(r)$ introduces exponentially decaying corrections to the pure inverse-power behavior at infinity but such corrections do not affect the polynomial tails. The numerical verification of the formulas (15), (22), and (28) is shown in Tables II and III. The observed decay rates agree perfectly with analytic predictions, while small errors in the amplitudes are due to (neglected) higher-order terms in the perturbation expansion.

VI. SCHWARZSCHILD BACKGROUND

Consider the evolution of the massless scalar field outside the $d + 1$ dimensional Schwarzschild black hole

TABLE II. The generic case: numerical verification of the analytic formula (15) for the potential (31a) ($\lambda = 0.1$) and initial data (29). Comparing the second column of this table (corresponding to $\alpha = 3.01$) with the last column of Table III one can see the discontinuity of the decay rate at $\alpha = 3$ (for $d = 5$ and 7).

		$\alpha = 2.5$		$\alpha = 3.01$		$\alpha = 4$	
		Theory	Numerics	Theory	Numerics	Theory	Numerics
$d = 3$	Exponent	2.5	2.499	3.01	3.009	4	4.00002
	Amplitude	−0.1253	−0.0881	−0.1785	−0.1518	−0.3545	−0.3320
$d = 5$	Exponent	4.5	4.501	5.01	5.0101	6	5.9999
	Amplitude	0.0261	0.0235	−0.000 89	−0.000 85	−0.2363	−0.2318
$d = 7$	Exponent	6.5	6.501	7.01	7.01	8	7.9999
	Amplitude	−0.0294	−0.0276	0.000 89	0.000 87	0.1418	0.1404

TABLE III. The exceptional case: comparison of analytic and numerical parameters of the tails for the potential (31b) (the first two columns) and (31a) (the third column) with $\alpha = 3$, $\lambda = 0.1$, and initial data (29). The analytic results are given by the formula (28) for the subtype IIa potential, and by the formula (27) for the type I and IIb potentials. Note that although the dominant tails for the type I and the subtype IIb potentials are theoretically the same, in the case IIb there is an additional first-order error due to the subdominant term $\mathcal{O}(\lambda)t^{-(2l+\alpha+\gamma)}$ which accounts for a slight difference in numerical accuracy between these two cases.

		$\gamma = 1/2$ (subtype IIa)		$\gamma = 1.75$ (subtype IIb)		(type I)	
		Theory	Numerics	Theory	Numerics	Theory	Numerics
$d = 5$	Exponent	5.5	5.4993	6	6.002	6	6.0000
	Amplitude	−0.0731	−0.0696	0.008 86	0.008 62	0.008 86	0.008 43
$d = 7$	Exponent	7.5	7.4998	8	8.0003	8	7.9999
	Amplitude	0.0603	0.0579	−0.001 77	−0.001 75	−0.001 77	−0.001 72
$d = 9$	Exponent	9.5	9.4999	10	9.9957	10	9.9997
	Amplitude	−0.1131	−0.1115	0.001 52	0.001 45	0.001 52	0.001 49

$$ds^2 = -\left(1 - \frac{1}{r^{d-2}}\right)dt^2 + \left(1 - \frac{1}{r^{d-2}}\right)^{-1} dr^2 + r^2 d\Omega^2_{d-1}, \tag{32}$$

where $d\Omega^2_{d-1}$ is the round metric on the unit sphere S^{d-1} and $d \geq 5$ is odd. Here we use units in which the horizon radius is at $r = 1$. Introducing the tortoise coordinate x, defined by $dr/dx = 1 - 1/r^{d-2}$, and decomposing the scalar field into multipoles, one obtains the following reduced wave equation for the jth multipole [14]:

$$\partial_t^2\psi - \partial_x^2\psi + U(x)\psi = 0,$$
$$U = \left(1 - \frac{1}{r^{d-2}}\right)\left(\frac{(2j + d - 3)(2j + d - 1)}{4r^2} + \frac{(d-1)^2}{4r^d}\right). \tag{33}$$

Note that (33) is the $1 + 1$ dimensional wave equation on the whole axis $-\infty < x < \infty$. For large positive x we have

$$r = x + \frac{1}{d-3}\frac{1}{x^{d-3}} - \frac{d-2}{(2d-5)(d-3)}\frac{1}{x^{2d-5}} + \mathcal{O}\left(\frac{1}{x^{3d-7}}\right), \tag{34}$$

which implies that

$$U(x) = \frac{(2j + d - 3)(2j + d - 1)}{4x^2} + V(x),$$
$$V(x) = \frac{a}{x^d} + \frac{b}{x^{2d-2}} + \mathcal{O}\left(\frac{1}{x^{3d-4}}\right) \quad \text{as } x \to \infty, \tag{35}$$

with

$$a = -\frac{(d-1)j(j + d - 2)}{d-3} \quad \text{and} \quad b = -\frac{(2d-3)((d-3)(d-2)^2(d-1) - 4j(j + d - 2)(1 + d(d-3)))}{4(2d-5)(d-3)^2}. \tag{36}$$

For large negative x (near the horizon) the potential is exponentially small, so one expects that the backscattering off the left edge of the potential can be neglected. If so, the decay rate (but not the amplitude) should follow from the analysis of Sec. IV. Comparing Eq. (33) for large positive x to Eq. (18) with the potential (25) and using (35) we find that $l = j + (d-3)/2$ and the potential V is of the subtype IIa with $\alpha = d$ and $\gamma = d - 2$. Thus, applying (28) we get the first-order tail

$$\psi(t, x) \sim t^{-(2j+3d-5)}. \tag{37}$$

This prediction has been verified numerically for $d = 5$.

Remark 3. Late-time tails outside higher-dimensional Schwarzschild black holes were studied in [9], however in the even-dimensional case the reasoning presented there is not correct, even though the result agrees with (37). The reason is that the analysis of [9] is based on the application of Ching *et al.*'s conjecture about the decay of the second-order tail $t^{-(2l+2\alpha-2)}$ which for $l = j + (d-3)/2$ and $\alpha = d$ gives $t^{-(2j+3d-5)}$. Unfortunately, this conjecture does not apply to the problem at hand. For $j = 0$ this is evident because the leading term in V (proportional to x^{-d}) vanishes (since by (36) $a = 0$), while the subleading term (proportional to $x^{-(2d-2)}$) is of generic type. For $j > 0$ this follows from the fact that the potential is of the subtype IIa. Thus, for all $j \geq 0$ the dominant (first-order) contribution to the tail comes from the subleading term in the potential. The agreement of the decay rate obtained in [9] with (37) is accidental and due to the fact that the subdominant term in (35) (not considered in [9]) is on a borderline between subtypes IIa and IIb.

Admittedly, the handwaving argument leading to (37) is far from satisfactory. Unfortunately, we have not been able to carry over the analysis from Secs. II, III, and IV in the case of Eq. (33). There are two difficulties in this respect. First, in contrast to the spherical case, Huygens' principle is not valid for the free wave equation in $1 + 1$ dimensions. Second, there is no natural small parameter in the problem. In the impressive *tour de force* work [7], Barack showed how to overcome these difficulties for a restricted class of initial data in four dimensions. It would be interesting to generalize Barack's approach to higher even-dimensional Schwarzschild spacetimes.

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APPENDIX

Throughout the appendix we use the notation of [10] in which the square bracket around a logical expression returns a value 1 if the expression is true and a value 0 if the expression is false:

$$[\text{condition}] = \begin{cases} 1 & \text{if condition} = \text{true} \\ 0 & \text{if condition} = \text{false} \end{cases}.$$

In order to derive the asymptotic behavior of the iterations (13) and (19) near timelike infinity (fixed r and $t \to \infty$) we need to evaluate the following expression:

$$\mathcal{F}(t, r; \beta, L) = -\frac{2^\beta}{4r^{l+1}} \sum_{k=0}^{L} c_{L,k} \int_{-\infty}^{+\infty} d\eta \int_{t-r}^{t+r} d\xi \frac{P_l(\mu)}{(\xi - \eta)^{\beta+L-k}} a^{(k)}(\eta), \tag{A1}$$

where

$$c_{L,k} = \frac{(2L-k)!}{k!(L-k)!} \tag{A2}$$

and

$$\mu = \frac{(\xi - t)(t - \eta) + r^2}{r(\xi - \eta)}. \tag{A3}$$

From (9) and (13) we have

$$\phi_1(t, r) = \mathcal{F}(t, r; \alpha, l), \tag{A4}$$

and from (19) and (21) we have

$$\phi_2(t, r) = \frac{1}{2(\alpha - 1)r^{\alpha+l}} \sum_{q=0}^{l-\alpha/2+1/2} (l - \alpha/2 + 1/2)^{\underline{q}} \cdot \frac{2^q (\alpha/2)^{\overline{q}}}{\alpha^{\overline{q}}} \mathcal{F}(t, r; 2\alpha - 1 + q, l - 1 - q). \tag{A5}$$

Since $a(\eta)$ has compact support, it is advantageous to begin with integration by parts,

$$\int_{-\infty}^{+\infty} d\eta \frac{P_l(\mu)}{(\xi-\eta)^{\beta+L-k}} a^{(k)}(\eta) = \int_{-\infty}^{+\infty} d\eta (-1)^k \frac{d^k}{d\eta^k}\left(\frac{P_l(\mu)}{(\xi-\eta)^{\beta+L-k}}\right) a(\eta).$$

For μ as defined in (A3) and for any function $g(\mu)$ the following identity holds:

$$\frac{d^k}{d\eta^k}\left(\frac{g(\mu)}{(\xi-\eta)^{\beta}}\right) = \sum_{j=0}^{k}\binom{k}{j}(\beta+k-1)^{\underline{k-j}}\left(\frac{r^2-(t-\xi)^2}{r}\right)^j \frac{g^{(j)}(\mu)}{(\xi-\eta)^{\beta+k+j}}, \tag{A6}$$

hence

$$\mathcal{F}(t,r;\beta,L) = -\frac{2^{\beta}}{4r^{l+1}}\int_{-\infty}^{+\infty} d\eta a(\eta) \sum_{0\le j\le k\le L}(-1)^k\binom{k}{j} c_{L,k}(\beta+L-1)^{\underline{k-j}}\frac{1}{r^j}\int_{t-r}^{t+r} d\xi \frac{(r^2-(t-\xi)^2)^j}{(\xi-\eta)^{\beta+L+j}} P_l^{(j)}(\mu). \tag{A7}$$

The sum over k can be evaluated explicitly:

$$\sum_{k=j}^{L}(-1)^k\binom{k}{j}\frac{(2L-k)!}{k!(L-k)!}(\beta+L-1)^{\underline{k-j}} = (-1)^L\binom{L}{j}(\beta-2)^{\underline{L-j}}. \tag{A8}$$

Let us define

$$\mathcal{I} := \frac{1}{r^j}\int_{t-r}^{t+r} d\xi \frac{(r^2-(t-\xi)^2)^j}{(\xi-\eta)^{\beta+L+j}} P_l^{(j)}(\mu). \tag{A9}$$

Changing the integration variable from ξ to μ and integrating by parts, we get

$$\mathcal{I} = (-1)^j \frac{r^{j+1}(t-\eta)^{\beta-2+L-j}}{[(t-\eta)^2-r^2]^{\beta-1+L}}\int_{-1}^{+1} d\mu P_l(\mu)\frac{d^j}{d\mu^j}\left[(1-\mu^2)^j\left(1-\frac{r}{t-\eta}\mu\right)^{\beta-2+L-j}\right]. \tag{A10}$$

Using the identity [15]

$$\mu^k = \sum_{l=k,k-2,k-4,\ldots}\frac{(2l+1)k!}{2^{(k-l)/2}(\frac{k-l}{2})!(k+l+1)!!}P_l(\mu), \tag{A11}$$

and expanding $\frac{d^j}{d\mu^j}[(1-\mu^2)^j(1-\frac{r}{t-\eta}\mu)^{\beta-2+L-j}]$ in Taylor series we get

$$\begin{aligned}
\mathcal{I} &= (-1)^j \frac{r^{j+1}(t-\eta)^{\beta-2+L-j}}{[(t-\eta)^2-r^2]^{\beta-1+L}}\sum_{n=0}^{\beta-2+L}(j+n)^{\underline{j}}\int_{-1}^{+1} d\mu P_l(\mu)\mu^n \\
&\quad\times \sum_{m=0}^{[(j+n)/2]}\binom{j}{m}\binom{\beta-2+L-j}{j+n-2m}(-1)^{j+n+m}\left(\frac{r}{t-\eta}\right)^{j+n-2m} \\
&= \frac{r^{l+1}(t-\eta)^{\beta-2+L-l}}{[(t-\eta)^2-r^2]^{\beta-1+L}}\sum_{n=0}^{[(\beta-2+L-l)/2]}(j+l+2n)^{\underline{j}}\int_{-1}^{+1} d\mu P_l(\mu)\mu^{l+2n} \\
&\quad\times \sum_{m=0}^{[(j+l+2n)/2]}\binom{j}{m}\binom{\beta-2+L-j}{j+l+2n-2m}(-1)^{l+m}\left(\frac{r}{t-\eta}\right)^{2j+2n-2m} \\
&= \frac{r^{l+1}(t-\eta)^{\beta-2+L-l}}{[(t-\eta)^2-r^2]^{\beta-1+L}}\sum_{n=0}^{[(\beta-2+L-l)/2]}(j+l+2n)^{\underline{j}}2^{l+1}\frac{(l+2n)!(l+n)!}{n!(2l+2n+1)!} \\
&\quad\times \sum_{m=0}^{[(j+l+2n)/2]}\binom{j}{m}\binom{\beta-2+L-j}{j+l+2n-2m}(-1)^{l+m}\left(\frac{r}{t-\eta}\right)^{2j+2n-2m}.
\end{aligned} \tag{A12}$$

Collecting the results of (A8), (A10), and (A12) and plugging them into (A7) we get

$$\mathcal{F}(t,r;\beta,L) = -\frac{2^{\beta+l+1}}{4}\int_{-\infty}^{+\infty} d\eta a(\eta)\frac{(t-\eta)^{\beta-2+L-l}}{[(t-\eta)^2-r^2]^{\beta-1+L}}\sum_{n=0}^{\lfloor(\beta-2+L-l)/2\rfloor}\frac{(l+2n)!(l+n)!}{n!(2l+2n+1)!}(-1)^{L+l}L!S(\beta,L), \quad (A13)$$

where

$$S(\beta,L) = \sum_{j=0}^{L}\binom{\beta-2}{L-j}\binom{j+l+2n}{j}\sum_{m=0}^{\lfloor(j+l+2n)/2\rfloor}(-1)^m\binom{j}{m}\binom{\beta-2+L-j}{j+l+2n-2m}\left(\frac{r}{t-\eta}\right)^{2j+2n-2m}. \quad (A14)$$

1. First-order approximation

To evaluate the first iteration $\phi_1(t,r)$ we apply the formula (A13) with $\beta=\alpha$ and $L=l$. Then

$$S(\alpha,l) = \sum_{j=0}^{l}\binom{\alpha-2}{l-j}\binom{l+2n+j}{j}\sum_{m=(j-l)/2}^{j+n}(-1)^{j+n-m}\binom{j}{j+n-m}\binom{\alpha-2+l-j}{l-j+2m}\left(\frac{r}{t-\eta}\right)^{2m}, \quad (A15)$$

where we shifted the summation index $m\to j+n-m$. Next, we interchange the order of summation according to

$$\begin{aligned}&[0\le j][j\le l][m-n\le j][j\le 2m+l]\\ &\Leftrightarrow\left[-\frac{l}{2}\le m<0\right][0\le j\le l+2m]+[0\le m<n][0\le j\le l]+[n\le m\le l+n][m-n\le j\le l],\end{aligned}$$

and convert the sum over j into a generalized hypergeometric function [10]. Defining

$$t_j = (-1)^{j+n-m}\binom{\alpha-2}{l-j}\binom{l+2n+j}{j}\binom{j}{j+n-m}\binom{\alpha-2+l-j}{l-j+2m},$$

we see that $t_0\neq 0$ iff $n=m$; thus the sums for $[-\frac{l}{2}\le m<0]$ and $[0\le m<n]$ do not contribute to (A15) and we are left with

$$S(\alpha,l) = \sum_{m=n}^{n+l}\left(\frac{r}{t-\eta}\right)^{2m}\sum_{j=0}^{l+n-m}(-1)^j\binom{\alpha-2}{l+n-m-j}\binom{l+n+m+j}{j+m-n}\binom{j+m-n}{j}\binom{\alpha-2+l+n-m-j}{l+n+m-j}, \quad (A16)$$

where we shifted the summation index $j\to j+m-n$. Defining

$$\tilde{t}_j = (-1)^j\binom{\alpha-2}{l+n-m-j}\binom{l+n+m+j}{j+m-n}\binom{j+m-n}{j}\binom{\alpha-2+l+n-m-j}{l+n+m-j},$$

we see that

$$\tilde{t}_0 = \frac{(\alpha-2)^{\underline{l+n-m}}}{(l+n-m)!}\cdot\frac{(l+n+m)!}{(m-n)!(l+2n)!}\cdot\frac{(\alpha-2+l+n-m)^{\underline{l+n+m}}}{(l+n+m)!}$$

and

$$\frac{\tilde{t}_{j+1}}{\tilde{t}_j} = \frac{(j-(l+n-m))(j-(l+n+m))(j+(l+n+m+1))}{(j+((\alpha-1)-(l+n-m)))(j+(-(\alpha-2)-(l+n-m)))(j+1)},$$

hence

$$\begin{aligned}
S(\alpha, l) &= \sum_{m=n}^{n+l}\left(\frac{r}{t-\eta}\right)^{2m}\frac{(\alpha-2)^{\underline{l+n-m}}}{(l+n-m)!}\cdot\frac{(\alpha-2+l+n-m)^{\underline{l+n+m}}}{(m-n)!(l+2n)!}\\
&\quad\times F\left(\begin{matrix}-(l+n-m), -(l+n+m), (l+n+m+1)\\ (\alpha-1)-(l+n-m), -(\alpha-2)-(l+n-m)\end{matrix}\,\middle|\, 1\right)\\
&= \sum_{m=n}^{n+l}\left(\frac{r}{t-\eta}\right)^{2m}2^{1+2(l+n-m)}\pi\frac{(\alpha-2)^{\underline{l+n-m}}}{(l+n-m)!}\cdot\frac{(\alpha-2+l+n-m)^{\underline{l+n+m}}}{(m-n)!(l+2n)!}\\
&\quad\times\frac{\Gamma(-(\alpha-2)-(l+n-m))\Gamma((\alpha-1)-(l+n-m))}{\Gamma(-\frac{\alpha-3}{2}+m)\Gamma(-\frac{\alpha-2}{2}-(l+n))\Gamma(\frac{\alpha}{2}+m)\Gamma(\frac{\alpha-1}{2}-(l+n))},
\end{aligned} \tag{A17}$$

where in the last equation we used the identity

$$F\left(\begin{matrix}a+1, -a, (b+c-1)/2\\ b, c\end{matrix}\,\middle|\, 1\right) = 2^{2-(b+c)}\pi\frac{\Gamma(b)\Gamma(c)}{\Gamma(\frac{b-a}{2})\Gamma(\frac{c-a}{2})\Gamma(\frac{1+b+a}{2})\Gamma(\frac{1+c+a}{2})}.$$

Substituting

$$(\alpha-2)^{\underline{l+n-m}}\Gamma((\alpha-1)-(l+n-m)) = \Gamma(\alpha-1)$$

and

$$(\alpha-2+l+n-m)^{\underline{l+n+m}}\Gamma(-(\alpha-2)-(l+n-m)) = (-1)^{l+n+m}\Gamma(-\alpha+2+2m)$$

into (A17), we get

$$S(\alpha, l) = \sum_{m=n}^{n+l}\left(\frac{r}{t-\eta}\right)^{2m}\frac{(-1)^{l+n+m}2^{1+2(l+n-m)}\pi}{(l+n-m)!(m-n)!(l+2n)!}\frac{\Gamma(\alpha-1)\Gamma(-\alpha+2+2m)}{\Gamma(\frac{\alpha}{2}+m)\Gamma(-\frac{\alpha-3}{2}+m)\Gamma(-\frac{\alpha-2}{2}-(l+n))\Gamma(\frac{\alpha-1}{2}-(l+n))}.$$

The last equation can be still simplified due to the identity

$$\frac{\Gamma(\alpha-1)\Gamma(-\alpha+2)}{\Gamma(\frac{\alpha}{2})\Gamma(-\frac{\alpha-3}{2})\Gamma(-\frac{\alpha-2}{2}-l)\Gamma(\frac{\alpha-1}{2}-l)} = \frac{(-1)^l}{2\pi}\left(\frac{\alpha-3}{2}\right)^{\underline{l}}\left(\frac{\alpha}{2}\right)^{\overline{l}}. \tag{A18}$$

We have

$$\begin{aligned}
\Gamma(-\alpha+2+2m) &= (-\alpha+2)^{\overline{2m}}\Gamma(-\alpha+2),\\
\Gamma\left(-\frac{\alpha-3}{2}+m\right) &= \left(-\frac{\alpha-3}{2}\right)^{\overline{m}}\Gamma\left(-\frac{\alpha-3}{2}\right),\\
\Gamma\left(-\frac{\alpha-2}{2}-l-n\right) &= \frac{\Gamma(-\frac{\alpha-2}{2}-l)}{(-\frac{\alpha-2}{2}-l-1)^{\underline{n}}},\\
\Gamma\left(\frac{\alpha-1}{2}-l-n\right) &= \frac{\Gamma(\frac{\alpha-1}{2}-l)}{(\frac{\alpha-1}{2}-l-1)^{\underline{n}}},\\
\Gamma\left(\frac{\alpha}{2}+m\right) &= \left(\frac{\alpha}{2}\right)^{\overline{m}}\Gamma\left(\frac{\alpha}{2}\right),
\end{aligned}$$

and

$$\frac{(-\alpha+2)^{\overline{2m}}}{(-\frac{\alpha-3}{2})^{\overline{m}}} = 2^{2m}\left(-\frac{\alpha}{2}+1\right)^{\overline{m}},$$

so finally

$$S(\alpha, l) = \sum_{m=n}^{n+l} \left(\frac{r}{t-\eta}\right)^{2m} \frac{(-1)^{n+m} 2^{2(l+n)}}{(l+n-m)!(m-n)!(l+2n)!} \left(\frac{\alpha-3}{2}\right)^{\underline{l}} \left(\frac{\alpha}{2}\right)^{\overline{l}} \frac{(-\frac{\alpha}{2}+1)^{\overline{m}} (-\frac{\alpha-2}{2} - l - 1)^{\underline{n}} (\frac{\alpha-1}{2} - l - 1)^{\underline{n}}}{(\frac{\alpha}{2})^{\overline{m}}}. \tag{A19}$$

Plugging (A19) into (A13) with $\beta = \alpha$ and $L = l$ we get the expression (14).

2. Second-order approximation

The calculation in the second order ($\beta = 2\alpha - 1 + q$ and $L = l - 1 - q$) is only a slight modification of what we have already done in the first order. Following the same steps which led us from (A15) to (A18) we get

$$S(\beta, L) = \sum_{m=n}^{n+L} \left(\frac{r}{t-\eta}\right)^{2m} \frac{(-1)^{l+n+m} 2^{1+2(L+n-m)} \pi}{(L+n-m)!(m-n)!(l+2n)!} \times \frac{\Gamma(\beta-1)\Gamma(-\beta+2+l-L+2m)}{\Gamma(\frac{\beta}{2} + \frac{l-L}{2} + m)\Gamma(-\frac{\beta-3}{2} + \frac{l-L}{2} + m)\Gamma(-\frac{\beta-2}{2} - (\frac{l+L}{2} + n))\Gamma(\frac{\beta-1}{2} - (\frac{l+L}{2} + n))}. \tag{A20}$$

The last equation can be simplified due to the identity

$$\frac{\Gamma(\beta-1)\Gamma(-\beta+2+l-L)}{\Gamma(\frac{\beta}{2} + \frac{l-L}{2})\Gamma(-\frac{\beta-3}{2} + \frac{l-L}{2})\Gamma(-\frac{\beta-2}{2} - \frac{l+L}{2})\Gamma(\frac{\beta-1}{2} - \frac{l+L}{2})} = \frac{(-1)^l}{2\pi} \left(\frac{\beta - 3 - (l-L)}{2}\right)^{\underline{L}} \left(\frac{\beta + l - L}{2}\right)^{\overline{L}} (\beta-2)^{\underline{l-L}}, \tag{A21}$$

which for $L = l$ reduces to (A18). We have

$$\begin{aligned}
\Gamma(-\beta+2+l-L+2m) &= (-\beta+2+l-L)^{\overline{2m}}\Gamma(-\beta+2+l-L),\\
\Gamma\left(-\frac{\beta-3}{2} + \frac{l-L}{2} + m\right) &= \left(-\frac{\beta-3}{2} + \frac{l-L}{2}\right)^{\overline{m}} \Gamma\left(-\frac{\beta-3}{2} + \frac{l-L}{2}\right),\\
\Gamma\left(-\frac{\beta-2}{2} - \frac{l+L}{2} - n\right) &= \frac{\Gamma(-\frac{\beta-2}{2} - \frac{l+L}{2})}{(-\frac{\beta-2}{2} - \frac{l+L}{2} - 1)^{\underline{n}}},\\
\Gamma\left(\frac{\beta-1}{2} - \frac{l+L}{2} - n\right) &= \frac{\Gamma(\frac{\beta-1}{2} - \frac{l+L}{2})}{(\frac{\beta-1}{2} - \frac{l+L}{2} - 1)^{\underline{n}}},\\
\Gamma\left(\frac{\beta}{2} + \frac{l-L}{2} + m\right) &= \left(\frac{\beta}{2} + \frac{l-L}{2}\right)^{\overline{m}} \Gamma\left(\frac{\beta}{2} + \frac{l-L}{2}\right),
\end{aligned}$$

and

$$\frac{(-\beta+2+l-L)^{\overline{2m}}}{(-\frac{\beta-3}{2} + \frac{l-L}{2})^{\overline{m}}} = 2^{2m}\left(-\frac{\beta}{2} + \frac{l-L}{2} + 1\right)^{\overline{m}},$$

hence

$$S(\beta, L) = \sum_{m=n}^{n+L} \left(\frac{r}{t-\eta}\right)^{2m} \frac{(-1)^{n+m} 2^{2(L+n)}}{(L+n-m)!(m-n)!(l+2n)!} \left(\frac{\beta - 3 - (l-L)}{2}\right)^{\underline{L}} \left(\frac{\beta + l - L}{2}\right)^{\overline{L}} (\beta-2)^{\underline{l-L}} \times \frac{(-\frac{\beta}{2} + \frac{l-L}{2} + 1)^{\overline{m}} (-\frac{\beta-2}{2} - \frac{l+L}{2} - 1)^{\underline{n}} (\frac{\beta-1}{2} - \frac{l+L}{2} - 1)^{\underline{n}}}{(\frac{\beta}{2} + \frac{l-L}{2})^{\overline{m}}}. \tag{A22}$$

Plugging (A22) into (A13) we get

$$\begin{aligned}\mathcal{F}(t,r;2\alpha-1+q,l-1-q) = {} & (-1)^q \frac{2^{2\alpha+3l-2-q}}{4}\left(\alpha-\frac{5}{2}\right)^{\overline{l-1-q}} \\ & \times(\alpha-2+l)^{\underline{l-1-q}}(2\alpha-3)^{\overline{1+q}}\int_{-\infty}^{+\infty} d\eta a(\eta)\frac{(t-\eta)^{2\alpha-4}}{[(t-\eta)^2-r^2]^{2\alpha-3+l}} \\ & \times\sum_{n=0}^{\alpha-2}(-1)^n\frac{2^{2n}(l+n)!}{n!(2l+2n+1)!}(-\alpha+1-l)^{\underline{n}}\left(\alpha-\frac{3}{2}-l+q\right)^{\underline{n}} \\ & \times\sum_{m=n}^{n+l-1-q}(-1)^m\binom{l-1-q}{m-n}\frac{(-\alpha+2)^{\overline{m}}}{(\alpha+q)^{\overline{m}}}\left(\frac{r}{t-\eta}\right)^{2m}. \end{aligned} \tag{A23}$$

Substituting this into (A5) and expanding in $1/t$ we have

$$\begin{aligned}\phi_2(t,r) = {} & \frac{1}{2(\alpha-1)}\cdot\frac{2^{2\alpha+2l-2}}{4(2l+1)!!}\cdot\frac{1}{t^{2\alpha+2l-2}}\left[A+2(\alpha+l-1)\frac{B}{t}+\mathcal{O}\left(\frac{1}{t^2}\right)\right] \\ & \times\left(\sum_{q=0}^{l-(\alpha-1)/2}(-1)^q(l-p)^{\underline{q}}\frac{2^q(\alpha/2)^{\overline{q}}}{\alpha^{\overline{q}}}\left(\alpha-\frac{5}{2}\right)^{\overline{l-1-q}}(\alpha-2+l)^{\underline{l-1-q}}(2\alpha-3)^{\overline{1+q}}\right), \end{aligned} \tag{A24}$$

with A and B defined in (17). Converting the sum over q into the generalized hypergeometric function we get (22).

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