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Shrinkers, expanders, and the unique continuation beyond generic blowup in the heat flow for harmonic maps between spheres

Paweł Biernat and Piotr Bizoń

M. Smoluchowski Institute of Physics, Jagiellonian University, Kraków, Poland

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Abstract

Using mixed analytical and numerical methods we investigate the development of singularities in the heat flow for corotational harmonic maps from the d-dimensional sphere to itself for $3 \leqslant d \leqslant 6$. By gluing together shrinking and expanding asymptotically self-similar solutions we construct global weak solutions which are smooth everywhere except for a sequence of times $T_1 < T_2 < \ldots < T_k < \infty$ at which there occurs the type I blow-up at one of the poles of the sphere. We give evidence that in the generic case the continuation beyond blow-up is unique, the topological degree of the map changes by one at each blow-up time T_i, and eventually the solution comes to rest at the zero energy constant map.

Mathematics Subject Classification: 58E20, 58J35, 35C06

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Let M and N be Riemannian manifolds with metric tensors g_{ij} and G_{AB} in some local coordinates $\{x^i\}$ and $\{X^A\}$. A map $X : M \to N$ is called harmonic if it is a critical point of the energy

$$E(X) = \int_M e(X)\sqrt{g}\,\mathrm{d}x, \qquad e(X) = \frac{1}{2}\frac{\partial X^A}{\partial x^i}\frac{\partial X^B}{\partial x^j}G_{AB}\,g^{ij}. \tag{1}$$

In this paper we consider harmonic maps from the d-dimensional unit sphere to itself, i.e. $M = N = S^d$ with g_{ij} and G_{AB} being standard round metrics. We parametrize S^d by spherical coordinates (θ, ϕ), where θ is colatitude $(0 \leqslant \theta \leqslant \pi)$ and ϕ is a point on the equator

S^{d-1} of S^d. We restrict our attention to corotational maps of the form $(\theta, \phi) \to (U(\theta), \phi)$. For such maps we have

$$E(U) = \frac{1}{2}\int_0^{\pi} \left(U_\theta^2 + (d-1)\,\frac{\sin^2 U}{\sin^2\theta} \right) \sin^{d-1}\theta \,\mathrm{d}\theta, \tag{2}$$

where for convenience we dropped the multiplicative factor $vol(S^{d-1})$ coming from the integration over ϕ. The Euler–Lagrange equation corresponding to the energy (2) reads

$$\frac{1}{\sin^{d-1}\theta}\left(\sin^{d-1}\theta\, U_\theta\right)_\theta - \frac{d-1}{2}\frac{\sin(2U)}{\sin^2\theta} = 0. \tag{3}$$

It was shown in [1] that for $3 \leqslant d \leqslant 6$ equation (3) has a countable sequence $\{U_n\}$ of smooth solutions of degree zero and one. These solutions may be viewed as excitations of the ground states: the constant map $U_0 = 0$ and the identity map $U_1 = \theta$, for even and odd values of n, respectively (for $d \geqslant 7$ these excitations disappear). For $n \to \infty$ the solutions $U_n(\theta)$ converge (nonuniformly) to the (singular) equator map $U_\infty = \pi/2$. Later, Corlette and Wald [2] rederived and extended these results using Morse theory methods. Their approach helped us to identify the two key features which are responsible for the existence of infinitely many solutions: the presence of the antipodal reflection symmetry $U \to \pi - U$ and the existence of the singular map $U_\infty = \pi/2$ of infinite index which is invariant under this symmetry. An essential ingredient of the Morse theoretic argument is an energy decreasing flow in the space of maps. In [2] this flow was defined in a somewhat *ad hoc* manner to ensure that it has all the desired technical properties. One might wonder if it is possible to repeat the Corlette–Wald argument using the ordinary heat flow. This would be interesting, for instance, in numerical implementations of the argument for similar systems. The main technical difficulty is that the heat flow can develop singularities in finite time. If this happens, in order to save the argument, one must find a way to continue the flow past a singularity in a unique manner. Although an analysis of this issue was the original motivation for this paper, the problem of uniqueness of continuation beyond blow-up in the heat flow for harmonic maps seems interesting in its own right, regardless of possible applications to elliptic problems.

The aim of this paper is two-fold. First, we describe the precise asymptotics of blow-up in the heat flow for corotational harmonic maps from the d-dimensional sphere to itself for $3 \leqslant d \leqslant 6$. We show that blow-up has the form of a shrinking self-similar solution (shrinker, for short). In turns out that among infinitely many shrinkers (whose existence was proved by Fan [3]), there is exactly one which is linearly stable. We provide numerical evidence that this stable shrinker determines the generic profile of blow-up. Second, we continue the flow past the singularity by gluing a suitable expanding self-similar solution (expander, for short). We find that there is exactly one expander which can be glued to the stable shrinker and consequently the continuation beyond the generic blow-up is unique.

The scenario of incomplete blow-up and self-similar global 'peaking solutions', that is solutions which shrink self-similarly, blow up, and then expand self-similarly for a while (with this scenario possibly repeating a number of times) has been studied in the past for the harmonic map flow [4] and other parabolic equations: the semilinear heat equations [5, 6], the mean curvature flow [4, 7–9], the Yang–Mills flow [10], the Ricci flow [11] and more recently for a fourth-order reaction-diffusion equation [12]. Most of these studies emphasized nonuniqueness of continuation beyond blow-up. To the best of our knowledge, this is the first work which demonstrates (by heuristic and numerical means) that for the generic blow-up the continuation is unique. As we shall see below, the uniqueness of continuation is contingent upon certain *quantitative* properties of self-similar solutions and thus may be hard to prove.

The rest of the paper is organized as follows. In section 2 we introduce the heat flow for equivariant harmonic maps from S^d (or $\mathbb{R}^d$) into S^d and recall basic facts about blow-up.

Section 3 is devoted to self-similar solutions of the heat flow for harmonic maps from $\mathbb{R}^d$ to S^d. Using matched asymptotics we derive asymptotic scaling formulae for the parameters of self-similar solutions. In section 4 we analyse the linear stability of self-similar solutions. In section 5 we study the continuation beyond blow-up and formulate the main result of this paper, that is the conjecture about the uniqueness of continuation in the generic case. Numerical evidence supporting this conjecture is presented in section 6. Finally, in section 7 we indicate possible extensions of our results.

2. Preliminaries

We consider the heat flow equation

$$U_t = \frac{1}{\sin^{d-1}\theta}\left(\sin^{d-1}\theta\, U_\theta\right)_\theta - \frac{d-1}{2}\frac{\sin(2U)}{\sin^2\theta}, \tag{4}$$

with initial and boundary conditions

$$U(0,\theta) = h(\theta) \in C^\infty[0,\pi], \tag{5}$$
$$U(t,0) = h(0) = 0, \tag{6}$$
$$U(t,\pi) = h(\pi) = k\pi, \tag{7}$$

where an integer k is the topological degree of the map. As long as the flow is smooth, the solution remains in the given homotopy class (i.e. the degree k does not change). It follows from (4) that for a smooth solution there holds

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\int_0^\pi U_t^2 \sin^{d-1}\theta\, \mathrm{d}\theta, \tag{8}$$

which shows that equation (4) is the gradient flow for the energy (2). Thus, one might expect that for $t \to \infty$ the solution $U(t,\theta)$ will converge to a critical point of E, i.e. a harmonic map. Unfortunately, as mentioned in the introduction, this expectation is too naive because in general the flow develops singularities in finite time. Indeed, it follows from general results for harmonic maps between compact manifolds that for any initial map with nonzero degree and sufficiently small energy the solution must blow up in finite time (see theorem 1.12 in [13]).

For equation (4), by symmetry, the singularity must occur at one of the poles. Since the blow-up is a localized phenomenon, the curvature of the domain manifold plays no role in the description of asymptotics of blow-up. Thus, from here until section 6 we replace the domain S^d by its tangent space at the pole, $\mathbb{R}^d$, and consider the heat flow for corotational harmonic maps from $\mathbb{R}^d$ to S^d:

$$u_t = \frac{1}{r^{d-1}}\left(r^{d-1}u_r\right)_r - \frac{d-1}{2r^2}\sin(2u), \tag{9}$$

where $u = u(t,r)$ $(r = |x|)$. Such maps enjoy scale invariance: if $u(t,r)$ is a solution, so is $u_\lambda(t,r) = u(t/\lambda^2, r/\lambda)$ for any positive number λ. Solutions which are invariant under rescaling, that is $u_\lambda = u$, are called self-similar. The self-similar solutions play the key role in the dynamics of type I blow-up[1] so the next three sections are devoted to their existence and properties.

Throughout the rest of this paper we assume that $3 \leqslant d \leqslant 6$.

[1] It is customary to divide singularities into two types: a singularity for which $(T-t)|\nabla u|^2$ is bounded as $t \nearrow T$ is said to be of type I; otherwise it is said to be of type II.

3. Self-similar solutions

3.1. Shrinkers

Let us assume that a solution of equation (9) develops a type I singularity at $r = 0$ in a finite time T, i.e. $(T-t)u_r^2(t,0)$ is bounded as $t \nearrow T$. To describe the formation of the singularity it is convenient to introduce new variables

$$s = -\ln(T-t), \qquad y = \frac{r}{\sqrt{T-t}}, \qquad f(s,y) = u(t,r). \tag{10}$$

In these variables equation (9) takes the form

$$f_s = \frac{1}{\rho}\left(\rho f_y\right)_y - \frac{d-1}{2y^2}\sin(2f), \qquad \rho(y) = y^{d-1}\exp(-y^2/4). \tag{11}$$

This equation can be viewed as the gradient flow for the functional

$$\mathcal{E}(f) = \frac{1}{2}\int_0^\infty \left(f_y^2 + \frac{d-1}{y^2}\sin^2 f\right)\rho\,\mathrm{d}y. \tag{12}$$

We shall refer to $\mathcal{E}(f)$ as the conformal energy because it is the energy for maps from $(\mathbb{R}^d, \mathrm{e}^{-\frac{y^2}{2(d-2)}}\delta)$ to S^d. A simple calculation gives

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}s} = -\int_0^\infty f_s^2\,\rho\,\mathrm{d}y, \tag{13}$$

hence the conformal energy is monotonically decreasing. The assumption that the blow-up is of type I implies that f_y is uniformly bounded as $s \to \infty$, hence the flow must converge to a critical point of the conformal energy, that is a solution of the Euler–Lagrange equation $\delta\mathcal{E}(f) = 0$ for $f(y)$

$$f'' + \left(\frac{d-1}{y} - \frac{y}{2}\right)f' - \frac{d-1}{2y^2}\sin(2f) = 0. \tag{14}$$

Note that an endpoint of evolution cannot be the trivial solution $f = 0$ as this would contradict the occurrence of blow-up at time T. Thus, the study of type I blow-up reduces to the study of nonconstant solutions of equation (14). We shall call such solutions shrinkers.

Let us discuss now existence and properties of shrinkers. Regular solutions of equation (14) behave near $y = 0$ as follows:

$$f(y) = ay - \frac{a(4da^2 - 4a^2 - 3)}{12(2+d)}y^3 + \mathcal{O}(y^5), \tag{15}$$

where a is a free parameter. Regular solutions at infinity behave as

$$f(y) = \frac{\pi}{2} + b - \frac{(d-1)\sin(2b)}{2y^2} + \mathcal{O}(y^{-4}), \tag{16}$$

where b is a free parameter. Using a shooting method Fan [3] proved that for $3 \leqslant d \leqslant 6$ there is an infinite sequence of pairs (a_n, b_n) for which the local solutions (15) and (16) are smoothly connected by a globally regular solution $f_n(y)$. The integer index n denotes the number of intersections of the solution $f_n(y)$ with $\pi/2$ (see figure 1). As $n \to \infty$ the shrinkers converge (nonuniformly) to the equator map $f_\infty = \pi/2$ and correspondingly $E(f_n) \to E(f_\infty) = 2^{d-1}\Gamma(\frac{d-1}{2})$. Some quantitative characteristics of shrinkers are displayed in table 1.

From the shooting argument in [3] it follows that $a_n \to \infty$ and $b_n \to 0$ as $n \to \infty$. We shall now use this fact to describe the behaviour of shrinkers for large n (we follow here

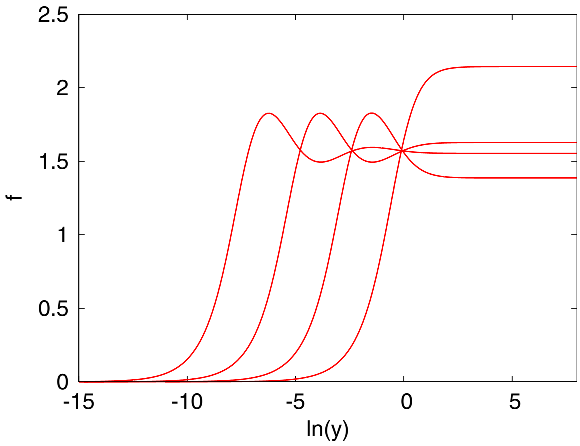

Figure 1. The profiles of the first four shrinkers for $d = 3$.

Table 1. Parameters of the first ten shrinkers for $d = 3$.

n	a_n	b_n	E_n
1	2.738753	0.573141	1.485688
2	2.927644×10^1	−0.184519	1.738165
3	3.141830×10^2	0.566142×10^{-1}	1.771588
4	3.376630×10^3	-0.172776×10^{-1}	1.776470
5	3.629513×10^4	0.527011×10^{-2}	1.778116
6	3.901390×10^5	-0.160744×10^{-2}	1.779706
7	4.193637×10^6	0.490287×10^{-3}	1.781650
8	4.507777×10^7	-0.149542×10^{-3}	1.784095
9	4.845449×10^8	0.456120×10^{-4}	1.787199
10	5.208415×10^9	-0.139121×10^{-4}	1.791128

a similar argument given in [1]). Let $\xi = ay$ and $\phi(\xi) = f(y)$. In terms of these variables equation (14) becomes

$$\phi'' + \left(\frac{d-1}{\xi} - \frac{\xi}{2a^2}\right)\phi' - \frac{d-1}{2\xi^2}\sin(2\phi) = 0 \tag{17}$$

with the initial condition $\phi(\xi) \sim \xi$ near $\xi = 0$. For $a \to \infty$, solutions of this equation tend uniformly on any compact interval to solutions of the limiting equation

$$\tilde{\phi}'' + \frac{d-1}{\xi}\tilde{\phi}' - \frac{d-1}{2\xi^2}\sin(2\tilde{\phi}) = 0 \tag{18}$$

with the same initial condition $\tilde{\phi}(\xi) \sim \xi$ near $\xi = 0$. Using the standard phase-plane analysis we obtain for $1 \ll \xi \ll a$

$$\tilde{\phi}(\xi) \simeq \frac{\pi}{2} + \alpha\, \xi^{-\frac{d-2}{2}} \sin(\omega \ln \xi + \delta), \qquad \omega = \frac{\sqrt{8d - d^2 - 8}}{2}, \tag{19}$$

where the amplitude α and the phase δ are uniquely determined by the initial condition $\tilde{\phi}'(0) = 1$. Returning to the original variables we obtain for $1/a \ll y \ll 1$

$$f(y) \simeq \frac{\pi}{2} + a^{-\frac{d-2}{2}} \alpha\, y^{-\frac{d-2}{2}} \sin(\omega \ln y + \omega \ln a + \delta). \tag{20}$$

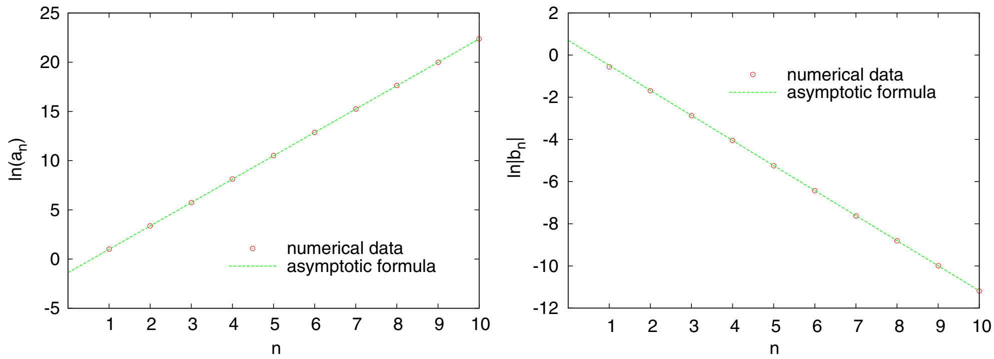

Figure 2. The asymptotic formulae (26) are shown to give excellent approximations for the parameters of shrinkers even for small n (here $d = 3$).

On the other hand, for $y \gg 1/a$ the solution is close to $\pi/2$ so we can write

$$f(y) \simeq \frac{\pi}{2} + b\, h(y), \tag{21}$$

where $h(y)$ is the solution of the linearized equation

$$h'' + \left(\frac{d-1}{y} - \frac{y}{2}\right) h' + \frac{d-1}{y^2}\, h = 0 \tag{22}$$

normalized by the condition $h(\infty) = 1$. For $1/a \ll y \ll 1$ we have

$$h(y) \simeq \alpha_1 y^{-\frac{d-2}{2}} \sin(\omega \ln y + \delta_1), \tag{23}$$

where α_1 and δ_1 are uniquely determined by the normalization condition $h(\infty) = 1$. Using (23) and matching the solutions (20) and (21) we obtain

$$a^{-\frac{d-2}{2}} \alpha \sin(\omega \ln y + \omega \ln a + \delta) \simeq b\, \alpha_1 \sin(\omega \ln y + \delta_1), \tag{24}$$

hence

$$\omega \ln a + \delta \simeq \delta_1 + n\pi, \qquad b \simeq (-1)^n \frac{\alpha}{\alpha_1} a^{-\frac{d-2}{2}}, \tag{25}$$

which yields the scaling laws for large n

$$a_n \simeq C \exp\left(\frac{n\pi}{\omega}\right), \qquad b_n \simeq (-1)^n D \exp\left(-\frac{n(d-2)\pi}{2\omega}\right), \tag{26}$$

where $C = \exp[(\delta_1 - \delta)/\omega]$ and $D = \frac{\alpha}{\alpha_1} C^{-\frac{d-2}{2}}$. Numerical parameters of shrinkers are displayed in table 1 and compared with the asymptotic expressions (26) in figure 2.

3.2. Expanders

To describe the behaviour of solutions for $t > T$ we introduce new variables

$$\sigma = \ln(t - T), \qquad y = \frac{r}{\sqrt{t - T}}, \qquad F(\sigma, y) = u(t, r), \tag{27}$$

in which equation (9) takes the form

$$F_\sigma = \frac{1}{R}\left(R\, F_y\right)_y - \frac{d-1}{2y^2} \sin(2F), \qquad R(y) = y^{d-1} \exp(y^2/4). \tag{28}$$

We shall refer to time-independent solutions of this equation as expanders. Equation (28) have been very recently studied by Germain and Rupflin [14] who established interesting results concerning existence, uniqueness and stability of expanders. Below we complement these results by a more detailed formal quantitative analysis (which is essential for our purposes).

Expanders satisfy the ordinary differential equation

$$F'' + \left(\frac{d-1}{y} + \frac{y}{2}\right) F' - \frac{d-1}{2y^2}\sin(2F) = 0 \tag{29}$$

with the regularity condition $F(y) \sim Ay$ near $y = 0$, where A is a free parameter. In contrast to shrinkers, expanders are globally regular for any A. This is due to the strong damping term $\frac{y}{2}F'$ in (29) which drives $F'(y)$ rapidly to zero as $y \to \infty$ and guarantees that $\lim_{y\to\infty} F(y)$ exists. Let $B = \lim_{y\to\infty} F(y) - \pi/2$. It is routine to show that B depends continuously on A. In order to obtain a more precise asymptotic behaviour, we rewrite equation (29) in the integral form

$$F'(y) = \frac{d-1}{2} y^{1-d} \mathrm{e}^{-y^2/4} \int_0^y s^{d-3} \mathrm{e}^{s^2/4} \sin(2F(s))\,\mathrm{d}s, \tag{30}$$

and compute the limit

$$\lim_{y\to\infty} y^3 F'(y) = (d-1) \lim_{y\to\infty} \frac{\int_0^y s^{d-3}\mathrm{e}^{s^2/4}\sin(2F(s))\,\mathrm{d}s}{2y^{d-4}\mathrm{e}^{y^2/4}} = -(d-1)\sin(2B), \tag{31}$$

where the last step follows from l'Hôpital's rule. Therefore, if $B \neq 0$, we have for large y

$$F(y) = \frac{\pi}{2} + B + \frac{d-1}{2y^2}\sin(2B) + \mathcal{O}(y^{-4}). \tag{32}$$

We note in passing that equation (44) below implies that there is an infinite countable subset of parameter values for which $B(A) = 0$ and

$$F(y) - \pi/2 \sim c\, y^{-d} \mathrm{e}^{-y^2/4} \qquad \text{as } y \to \infty. \tag{33}$$

The variational proof (using a renormalized energy) of existence of such rapidly decaying expanders was recently given in [14]. Since these solutions do not seem to participate in the dynamics of blow-up, we do not pursue them here in more detail.

Next, we derive asymptotic approximations of the function $B(A)$ for small and large arguments. For small A we substitute $F(y) = A\tilde{F}(y)$ into equation (29) and take the limit $A \to 0$ to obtain the linear equation

$$\tilde{F}'' + \left(\frac{d-1}{y} + \frac{y}{2}\right)\tilde{F}' - \frac{d-1}{y^2}\tilde{F} = 0 \tag{34}$$

with the initial condition $\tilde{F}(y) \sim y$ near $y = 0$. Clearly, the solution $\tilde{F}(y)$ is a positive monotonically increasing function converging to a constant at infinity. The explicit solution is

$$\tilde{F}(y) = y\mathrm{e}^{-\frac{y^2}{4}} M\left(\frac{d+1}{2}, \frac{d+2}{2}, \frac{y^2}{4}\right), \tag{35}$$

where $M(a, b, y)$ is the Kummer confluent hypergeometric function. Using the asymptotic expansion $M(a, b, x) \sim \frac{\Gamma(b)}{\Gamma(a)} x^{a-b}\mathrm{e}^x$ for large x [15] we obtain $\tilde{F}(\infty) = 2\Gamma(\frac{d+2}{2})/\Gamma(\frac{d+1}{2})$, thus for small A we have

$$B(A) \simeq -\frac{\pi}{2} + \frac{2\Gamma(\frac{d+2}{2})}{\Gamma(\frac{d+1}{2})} A. \tag{36}$$

For large A, repeating the argument leading to equation (20), we obtain for $1/A \ll y \ll 1$

$$F(y) \simeq \frac{\pi}{2} + A^{-\frac{d-2}{2}} \alpha\, y^{-\frac{d-2}{2}} \sin(\omega \ln y + \omega \ln A + \delta). \tag{37}$$

On the other hand, for $y \gg 1/A$ we can write

$$F(y) \simeq \frac{\pi}{2} + H(y), \tag{38}$$

where $H(y)$ is a solution of the linearized equation

$$H'' + \left(\frac{d-1}{y} + \frac{y}{2}\right) H' + \frac{d-1}{y^2} H = 0 \tag{39}$$

satisfying $H(\infty) = B$. In contrast to shrinkers, this normalization condition does not determine the solution uniquely since the two linearly independent solutions at infinity are

$$H_1(y) \sim 1 \qquad \text{and} \qquad H_2(y) \sim y^{-d} \exp(-y^2/4), \tag{40}$$

hence

$$H(y) = B H_1(y) + c\, H_2(y), \tag{41}$$

where c is an arbitrary constant. For $1/A \ll y \ll 1$ the solutions H_1 and H_2 behave as

$$H_i(y) \simeq C_i\, y^{-\frac{d-2}{2}} \sin(\omega \ln y + \Delta_i), \qquad i = 1, 2. \tag{42}$$

Combining equations (37), (41) and (42) we obtain the following matching condition:

$$A^{-\frac{d-2}{2}} \alpha\ \sin(\omega \ln y + \omega \ln A + \delta) \simeq B C_1 \sin(\omega \ln y + \Delta_1) + c\, C_2 \sin(\omega \ln y + \Delta_2), \tag{43}$$

which yields

$$B(A) \simeq \tilde{C} A^{-\frac{d-2}{2}} \sin(\omega \ln A + \tilde{\delta}), \tag{44}$$

where $\tilde{C}$ and $\tilde{\delta}$ are determined by $\alpha, \delta, C_i, \Delta_i$.

4. Linear stability of self-similar solutions

Now, we turn our attention to the linear stability analysis of shrinkers and expanders. The results of this analysis are important in understanding the dynamics of blow-up.

4.1. Shrinkers

Substituting $f(s, y) = f_n(y) + w(s, y)$ into equation (11) and retaining only linear terms in w, we obtain the evolution equation for linearized perturbations around the shrinker f_n

$$w_s = \frac{1}{\rho} \left(\rho w_y\right)_y - \frac{d-1}{y^2} \cos(2 f_n)\, w, \tag{45}$$

which after separation of variables, $w(s, y) = \mathrm{e}^{-\lambda s} v(y)$, yields the eigenvalue problem

$$\mathcal{A}_n v = \lambda v, \qquad \mathcal{A}_n = -\frac{1}{\rho} \partial_y \left(\rho \partial_y\right) + \frac{d-1}{y^2} \cos(2 f_n). \tag{46}$$

For each n the operator $\mathcal{A}_n$ is self-adjoint in the Hilbert space $X = L_2([0, \infty), \rho\, \mathrm{d}y)$. Both endpoints $y = 0$ and $y = \infty$ are of the limit-point type with admissible solutions behaving as $v(y) \sim y$ for $y \to 0$ and $v(y) \sim y^{2\lambda}$ for $y \to \infty$. Note that for each n there is an eigenvalue $\lambda = -1$ with the associated eigenfunction $v(y) = y f_n'(y)$. The presence of this gauge mode is due to time translation symmetry. To see this observe that if the blow-up time is shifted from T to $T + 2\varepsilon$, then

$$f(y) \to f\left(\frac{y}{\sqrt{1 + 2\varepsilon \mathrm{e}^s}}\right) = f(y) - \varepsilon \mathrm{e}^s y f'(y) + \mathcal{O}(\varepsilon^2). \tag{47}$$

Since $f_n'(y)$ has $(n-1)$ zeroes, it follows from the Sturm oscillation theorem that for the n-th shrinker there are exactly $(n-1)$ eigenvalues below -1. We checked numerically (but were unable to prove analytically) that there are no eigenvalues in the interval $-1 < \lambda \leqslant 0$. Denoting the spectrum by $\{\lambda_k^{(n)} | k = 0, 1, \ldots\}$ we thus have

$$\lambda_0^{(n)} < \lambda_1^{(n)} < \cdots < \lambda_{n-1}^{(n)} = -1 < 0 < \lambda_n^{(n)} < \cdots. \tag{48}$$

In conclusion, the shrinker f_n has exactly $(n-1)$ unstable modes (the gauge mode with $\lambda = -1$ is not counted as a genuine instability). In particular, the shrinker f_1 is linearly stable and therefore it is expected to participate in the generic dynamics of blow-up. This expectation will be confirmed numerically in section 6. The first few eigenvalues of the operator $\mathcal{A}_n$ for several n in $d = 3$ are displayed in table 2. Note that the columns in this table converge to limiting values, namely for each integer m we have

$$\lim_{n\to\infty} \lambda_{n+m}^{(n)} = \lambda_m. \tag{49}$$

Now, we will show that λ_m are the eigenvalues of the point spectrum of the operator

$$\mathcal{A}_\infty = -\frac{1}{\rho}\partial_y\left(\rho\partial_y\right) - \frac{d-1}{y^2}, \tag{50}$$

which is obtained from (46) by taking the (nonuniform) limit $f_n(y) \to \pi/2$ as $n \to \infty$. The potential term in (50) is unbounded from below as $y \to 0$ and $y = 0$ is a limit-circle point, so for $\mathcal{A}_\infty$ to be self-adjoint, we have to specify an additional boundary condition at $y = 0$ (which is usually referred to as the self-adjoint extension). This is done as follows. The solution of the eigenvalue equation $\mathcal{A}_\infty v = \lambda v$ which is admissible at infinity (i.e. behaving as $v(y) \sim y^{2\lambda}$ for $y \to \infty$) reads

$$v(y) = y^{1-\frac{d}{2}+\mathrm{i}\omega}\, U\left(\frac{1}{2} - \frac{d}{4} + \frac{\mathrm{i}\omega}{2} - \lambda, 1 + \mathrm{i}\omega, \frac{y^2}{4}\right), \tag{51}$$

where $U(a, b, z)$ is the Tricomi confluent hypergeometric function. Using the asymptotic expansion formula for $z \to 0$ (which is valid for $1 \leqslant \mathrm{Re}(b) < 2$) [15]

$$U(a, b, z) \sim \frac{\Gamma(1-b)}{\Gamma(a-b-1)} + \frac{\Gamma(b-1)}{\Gamma(a)}\, z^{1-b}, \tag{52}$$

we obtain from (51)

$$v(y) \sim y^{1-\frac{d}{2}} \cos\left(\omega \ln y + \Phi(\lambda)\right) \qquad \text{as } y \to 0, \tag{53}$$

where

$$\Phi(\lambda) = \arg\left(\frac{\Gamma(\mathrm{i}\omega)}{\Gamma\left(\frac{1}{2} - \frac{d}{4} + \frac{\mathrm{i}\omega}{2} - \lambda\right)}\right). \tag{54}$$

The self-adjoint extension amounts to fixing the phase $\Phi(\lambda)$ modulo π. A natural choice is to require that the eigenvalue $\lambda = -1$ belongs to the spectrum of $\mathcal{A}_\infty$. This leads to the quantization condition

$$\Phi(\lambda_{m-1}) = \Phi(-1) + m\pi, \qquad m \in \mathbb{Z}. \tag{55}$$

As shown in table 2, solutions of this equation, in fact, give the limit of the point spectra of the operators $\mathcal{A}_n$ for $n \to \infty$.

Table 2. The first few eigenvalues of the operator $\mathcal{A}_n$ in $d = 3$. Numerical solutions of the quantization condition (55) are listed in the last row.

n	$\lambda^{(n)}_{n-4}$	$\lambda^{(n)}_{n-3}$	$\lambda^{(n)}_{n-2}$	$\lambda^{(n)}_{n-1}$	$\lambda^{(n)}_{n}$	$\lambda^{(n)}_{n+1}$	$\lambda^{(n)}_{n+2}$
1				−1	0.517 62	1.630 38	2.696 84
2			−53.2995	−1	0.486 25	1.611 22	2.685 50
3		−6054.92	−52.4152	−1	0.482 71	1.608 79	2.683 80
4	−699 295	−5968.91	−52.3292	−1	0.482 37	1.608 58	2.683 63
⋮	⋮	⋮	⋮	−1	⋮	⋮	⋮
∞	−688 498	−5959.55	−52.3200	−1	0.482 34	1.608 52	2.683 61

4.2. Expanders

The linear stability analysis of expanders proceeds along the similar lines as above. Substituting $F(\sigma, y) = F(y) + W(\sigma, y)$ into equation (28) and linearizing we obtain the evolution equation for linearized perturbations around an expander $F(y)$

$$W_\sigma = \frac{1}{R}\left(R\, W_y\right)_y - \frac{d-1}{y^2}\cos(2F)\, W, \tag{56}$$

which after separation of variables, $W(\sigma, y) = \mathrm{e}^{-\Lambda\sigma} V(y)$, leads to the eigenvalue problem

$$\mathcal{B}V = \Lambda V, \qquad \mathcal{B} = -\frac{1}{R}\partial_y\left(R\,\partial_y\right) + \frac{d-1}{y^2}\cos(2F). \tag{57}$$

The operator $\mathcal{B}$ is self-adjoint in the Hilbert space $Y = L_2([0,\infty), R\,\mathrm{d}y)$. Both endpoints are of the limit-point type with admissible solutions $V(y) \sim y$ for $y \to 0$ and $V(y) \sim y^{2\lambda-d}\mathrm{e}^{-y^2/4}$ for $y \to \infty$. The gauge mode due to time translation symmetry $V(y) = yF'(y)$ has the (formal) eigenvalue $\Lambda = 1$, because if $T \to T + 2\varepsilon$, then

$$F(y) \to F\left(\frac{y}{\sqrt{1-2\varepsilon\mathrm{e}^{-\sigma}}}\right) = F(y) + \varepsilon\mathrm{e}^{-\sigma} yF'(y) + \mathcal{O}(\varepsilon^2). \tag{58}$$

The gauge mode is not an eigenfunction (because it does not belong to Y), nevertheless the Sturm oscillation theorem still applies and implies that an expander with n zeros of $F'(y)$ has exactly n eigenvalues below +1 (this was proved independently in [14]). In particular, monotonic expanders are linearly stable. Although this fact will be sufficient for the analysis of continuation beyond the generic blow-up, we wish to point out that using the 'turning-point' method [16] one can determine sharp stability intervals for expanders. This is done as follows. Let $F_A(y)$ denote the expander starting with $F'(0) = A$. Differentiating equation (29) with respect to A we find that $\frac{\partial F_A(y)}{\partial A}$ is the zero mode of the operator $\mathcal{B}$. In general, $\frac{\partial F_A(y)}{\partial A} \sim B'(A) \neq 0$ for $y \to \infty$; however, it follows from equation (44) that there is an increasing sequence of numbers A_k ($k \in N$) for which $B'(A_k) = 0$ and then, by equation (33)

$$\frac{\partial F_A(y)}{\partial A} \sim c'(A)\, y^{-d}\exp(-y^2/4), \tag{59}$$

hence the zero mode is a genuine eigenfunction. By [16] this implies that A_k are turning points at which the expander F_A picks a new unstable mode. More precisely, the expander F_A with $A \in (A_{k-1}, A_k)$ has exactly $(k-1)$ instabilities (here $A_0 = 0$ by definition).

5. Continuation beyond blow-up

Suppose that the solution of equation (9) develops a type I singularity at time T. Then, as we showed above, the profile of blow-up is given by one of the shrinkers

$$\lim_{t\nearrow T} u(t, r\sqrt{T-t}) = f_n(r), \qquad n \in \mathbb{N}. \tag{60}$$

In order to continue the solution beyond blow-up, for times $t > T$ we glue an expander which matches the shrinker f_n at time T, that is we require that

$$\lim_{t\searrow T} u(t, r\sqrt{t-T}) = F_A(r), \qquad F_A(\infty) = f_n(\infty) \Longleftrightarrow B(A) = b_n \tag{61}$$

or

$$\lim_{t\searrow T} u(t, r\sqrt{t-T}) = \pi - F_A(r), \qquad \pi - F_A(\infty) = f_n(\infty) \Longleftrightarrow B(A) = -b_n, \tag{62}$$

Note that in the case (61) the solution stays continuous across blow-up (hence the degree does not change), while in the case (62) the solution jumps at the origin from $u(t,0) = 0$ for $t < T$ to $u(t,0) = \pi$ for $t > T$ (hence the degree changes by one). In both cases we obtain a global weak solution which is smooth except for the time T.

Let $N(n)$ denote the number of roots of the equation $|B(A)| = |b_n|$. It follows from the large A formula for expanders (44) and large n formula for shrinkers (26) that $N(n)$ increases indefinitely with n. More precisely, we find numerically (see figure 3) that for $n \geqslant 2$

$$N(n) = \begin{cases} 2n-3 & \text{for } d = 3, 4, \\ 2n-1 & \text{for } d = 5, 6. \end{cases} \tag{63}$$

Since the shrinker f_n has $(n-1)$ instabilities, all $n \geqslant 2$ blow-ups are nongeneric phenomena of codimension $(n-1)$. It follows from the stability analysis of expanders that only one continuation is stable, namely that with $A_n^* = \min\{A : |B(A)| = |b_n|\}$. For this stable continuation the degree changes by one if n is odd and does not change if n is even.

Hereafter, we focus on the most important and interesting case $n = 1$ corresponding to the generic blow-up governed by the linearly stable shrinker f_1. In this case the equation $B(A) = b_1$ has no roots, while the equation $B(A) = -b_1$ has exactly one root (note that the existence of this root is guaranteed by the small A formula (36) and the continuity of the function $B(A)$), hence the continuation beyond blow-up is unique, stable, and changing degree. As this is our main result, let us phrase it in the form of a conjecture:

Conjecture 1. Let $3 \leqslant d \leqslant 6$. Suppose that $u(t,r)$ is a generic solution of equation (9) which develops a singularity at $r = 0$ in a finite time T. Then, for sufficiently small r there holds

$$u(t,r) \sim \begin{cases} f_1\left(\dfrac{r}{\sqrt{T-t}}\right) & \text{for } T - r^2 < t < T, \\ \dfrac{\pi}{2} + b_1 & \text{for } t = T, \\ \pi - F_{A_1^*}\left(\dfrac{r}{\sqrt{t-T}}\right) & \text{for } T < t < T + r^2, \end{cases} \tag{64}$$

where A_1^* is the (unique) root of equation $B(A) = -b_1$. This is illustrated in figure 4.

In the next section we present numerical evidence supporting this conjecture.

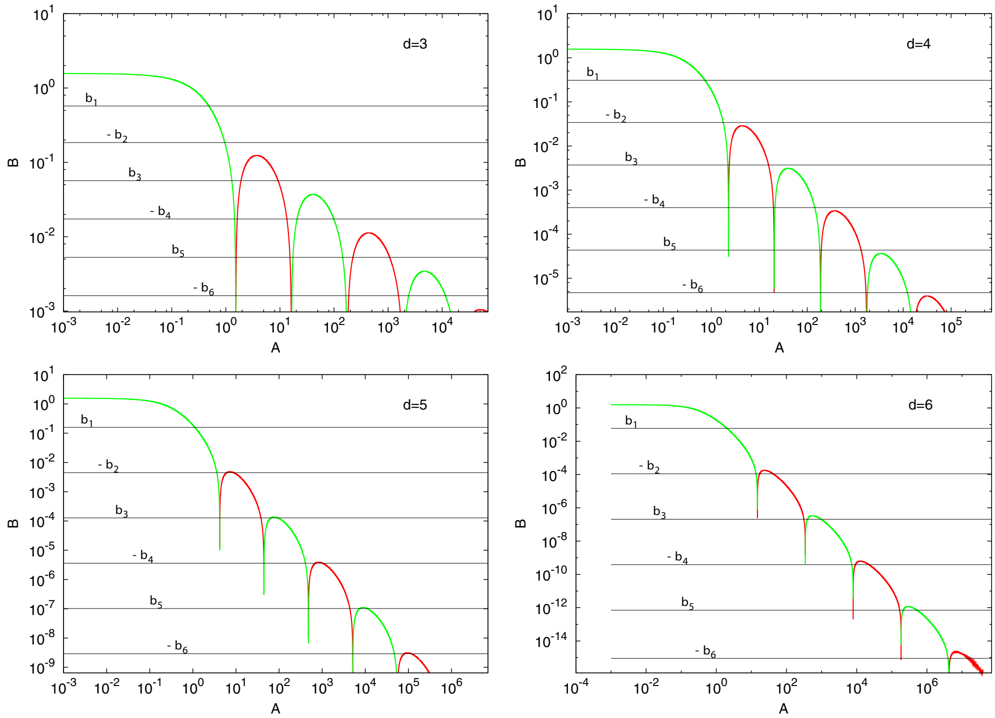

Figure 3. Plots of $|B(A)|$ in the log–log scale. The intersections with horizontal lines $|B| = |b_n|$ determine the number of continuations beyond blow-up.

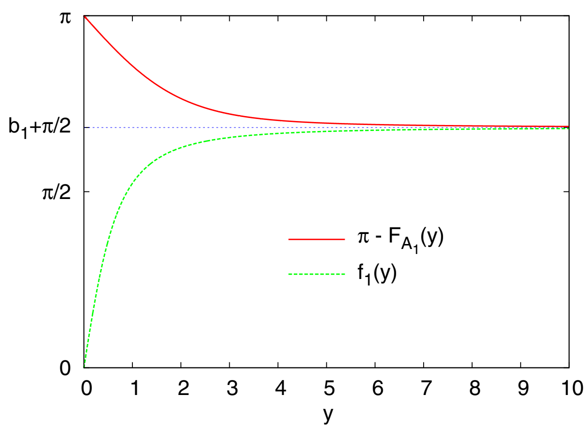

Figure 4. Gluing together the stable shrinker and expander in $d = 3$. Here $b_1 = 0.573\,141$ and $A_1^* = 0.483\,668$.

6. Numerical evidence

In this section we verify the above heuristic predictions by numerical simulations. In order to keep track of the structure of the singularity developing on a vanishingly small scale, it is necessary to use an adaptive method which refines the spatio-temporal grid near the singularity.

Our numerical method is based on the moving mesh method combined with the Sundman transformation, as described in [17], with some minor modifications and improvements specific to the problem at hand. This method is particularly efficient in computations of self-similar singularities. To implement the adaptivity in time we introduce a new computational time variable τ defined by

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = g(u), \qquad g(u) = \left|\frac{u_r}{u_{rt}}\right|_{r=0}. \tag{65}$$

Under this rescaling (called the Sundman transformation) the fixed time steps in τ correspond to $\Delta t_i \approx (T - t_i)\Delta\tau$ as $t \nearrow T$. In this way, the time scale of the developing singularity is identified automatically even though the blow-up time T is unknown beforehand. To implement the adaptivity in space we introduce a new computational spatial variable $\xi \in [0, 1]$ and define a mesh function $r(\xi, t)$ which places the moving mesh points at $r_i(t) = r(\mathrm{i}\Delta\xi, t)$. The function $r(\xi, t)$, whose role is to cluster the mesh points near the singularity, is determined by an auxiliary moving mesh partial differential equation (MMPDE), which is solved simultaneously with the original PDE. We use the so-called MMPDE6 [18]

$$\varepsilon r_{t\xi\xi} = -(M r_\xi)_\xi, \tag{66}$$

with the mesh density $M = |u_r| + \sqrt{|u_{rr}|}$ and the time-dependent relaxation parameter $\varepsilon(t) = 100\sqrt{g(t)} + 0.05$ (this $\varepsilon(t)$, found empirically, results in a better performance than the customarily used constant value).

The harmonic map heat equation $u_t = N(u)$, where $N(u)$ is the right-hand side of equation (4) or equation (9), is now be rewritten as the system

$$t_\tau = g(u), \tag{67}$$

$$u_\tau + r_\tau u_r = g(u)N(u) \tag{68}$$

$$\varepsilon r_{\tau\xi\xi} = -g(u)(M r_\xi)_\xi. \tag{69}$$

These equations are discretized using a 5-point finite difference scheme and integrated via the Embedded Runge–Kutta–Fehlberg (RKF45) method.

The numerical results are presented below for $d = 3$ as an illustration; the behaviour of solutions is qualitatively the same in all dimensions $3 \leqslant d \leqslant 6$. Since the dynamics of blow-up does not depend on the curvature of the domain, we first show simulations for equation (9), and only at the end we show simulations of multiple blow-ups for the spherical domain equation (4).

We begin by demonstrating the convergence to the stable shrinker. Figure 5 depicts snapshots from a typical evolution ending in a singularity. As the blow-up is approached, the solution is seen to converge to the profile of the stable shrinker f_1.

According to the linearized stability analysis the deviation of the solution from the stable shrinker is expected to have the following form near $r = 0$ for $t \nearrow T$ (to avoid notational clutter, hereafter we drop the superscript (1) on the eigenvalues and the eigenfunctions)

$$u(t, r) - f_1(y) \simeq \sum_{k=1}^{\infty} c_k (T - t)^{\lambda_k} v_k(y) = c_1 (T - t)^{\lambda_1} v_1(y) + \mathcal{O}\left((T - t)^{\lambda_2}\right), \tag{70}$$

where $y = r/\sqrt{T - t}$. To verify this prediction we proceed as follows. Differentiating (70) twice and using the normalization $v_1'(0) = 1$ we obtain

$$\partial_t \left[(T - t)^{1/2} u_r\right]\Big|_{r=0} = -c_1 \lambda_1 (T - t)^{\lambda_1} + \mathcal{O}\left((T - t)^{\lambda_2}\right). \tag{71}$$

Fitting the right-hand side of this equation to the numerically computed left-hand side, we obtain the coefficient c_1 and the eigenvalue λ_1 (see the left panel of figure 6). The fit gives

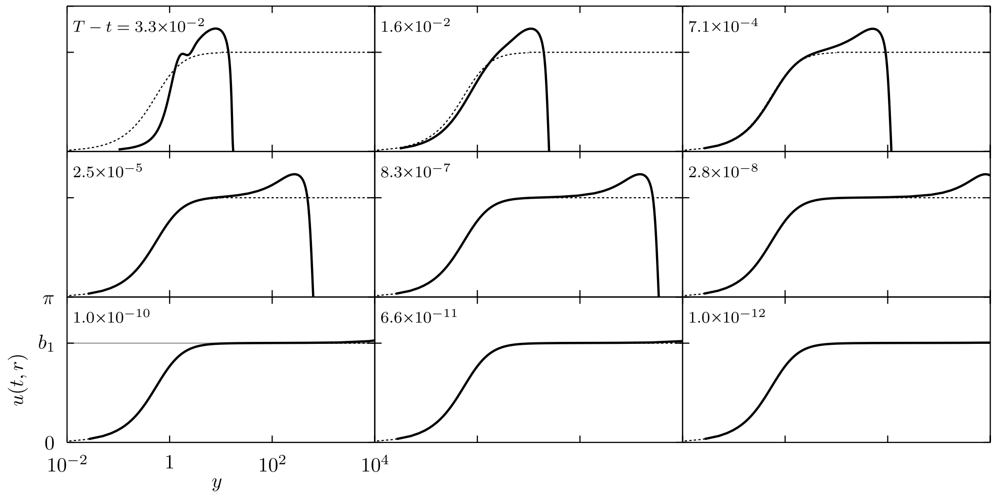

Figure 5. Convergence of the numerical solution (solid line) to the stable shrinker $f_1(y)$ (dotted line).

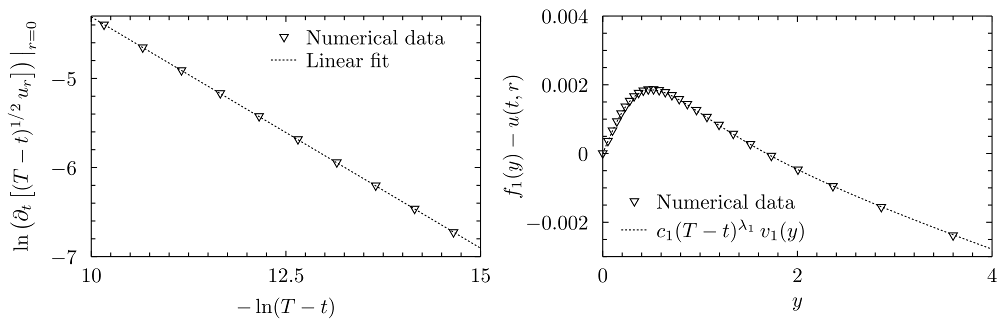

Figure 6. Left: The log–log plot of the left-hand side of expression (71). The linear fit gives $\lambda_1 = 0.519$. Right: we plot the deviation of the numerical solution from the stable shrinker at $T - t = 3.19 \times 10^{-6}$ and superimpose the first stable eigenmode $c_1(T-t)^{\lambda_1} v_1(y)$, obtained by solving the eigenvalue equation (46), with the coefficient c_1 taken from the fit in the left panel.

$\lambda_1 = 0.519$, in good agreement with the linearized stability analysis (see table 2). Next, in the right panel of figure 6 we show that near the blow-up time the left- and the right-hand sides of expression (70) (computed completely independently) do indeed agree.

Next, we describe the continuation beyond blow-up. In order to pass through the singularity we need to modify the numerical code. First, according to (62) we expect that at the blow-up time the solution is discontinuous at $r = 0$. This behaviour is not compatible with the boundary condition $u(t, 0) = 0$ implemented in our code. To go around this difficulty, we simply rewrite equation (9) in terms of $z(t, r) = ru(t, r)$ and impose the boundary condition $z(t, 0) = 0$ (which is compatible with the jump). In the case of equation (4) we use a similar trick introducing $Z(t, \theta) = \sin(\theta)\, U(t, \theta)$ as an independent variable. Second, at some late stage of blow-up (say, $T - t = 10^{-10}$) we must switch off the Sundman transformation since otherwise the time step would keep decreasing down to the machine precision, effectively freezing the simulation and preventing it to cross the time of blow-up. To this end, we replace $g(u)$ in (65) by $G(u) = g(u) + \Delta$ where $\Delta \approx 10^{-10}$ serves as a small scale cut-off. When

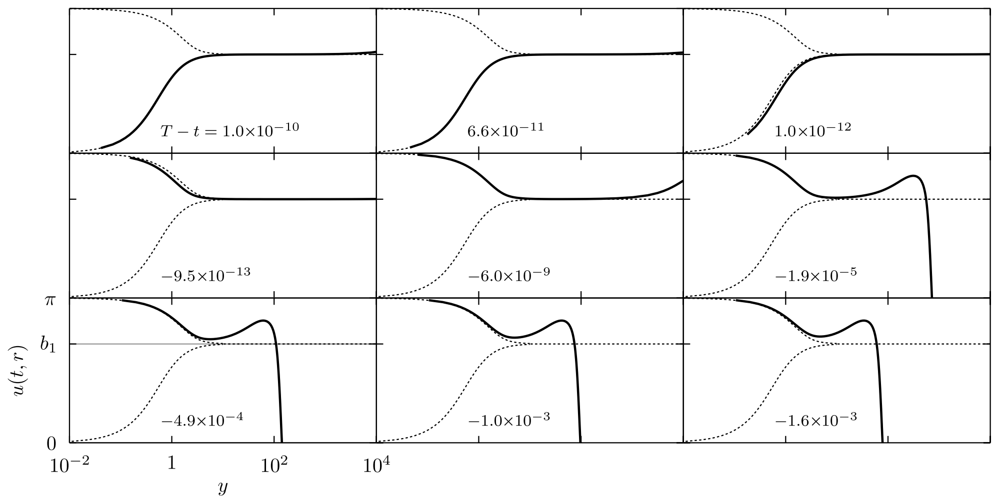

Figure 7. The same evolution as in figure 5 but using the modified numerical method which allows the solution to pass through the singularity. An additional dotted line shows the expander $\pi - F_{A_1^*}(y)$. Note that for $|T-t| \lesssim 10^{-11}$ the spatio-temporal resolution is lost and the numerical solution slightly deviates from the shrinker $f_1(y)$ (the third snapshot) and the expander $\pi - F_{A_1^*}(y)$ (the fourth snapshot). When the resolution is regained, the solution converges to the expander but later it moves away from it due to the interference with the far-field structure.

$g(u) \ll \Delta$, the solver loses its ability to adapt the time step appropriately and very quickly steps over the blow-up time. A moment afterwards, when $g(u)$ exceeds Δ again, the Sundman transformation is turned back on and keeps tracking of the, now growing, time scale of the expander. During a short time interval $T - 10^{-10} \lesssim t \lesssim T + 10^{-10}$ when the time adaptation procedure is suspended, the spatio-temporal scales are unresolved and the numerical solution is inaccurate (the third and the fourth snapshot in figure 7).

Applying this method, we continue the evolution shown in figure 5 past the singularity. In accordance with conjecture 1, almost immediately after the blow-up the numerical solution takes the form of the expander $\pi - F_{A_1^*}(y)$ (see figure 7). As written above, numerical evolution through a singularity necessarily involves an interval of uncontrolled behaviour due to the inevitable loss of resolution near the instant of blow-up. For this reason the simulation has limited reliability and taken alone would not provide ample evidence for the conjectured behaviour. It is the excellent consistency between numerics and the analytic insight, based on the understanding of self-similar solutions and their linear perturbations, which makes us feel confident that our conjecture is true.

Finally, let us consider the heat flow for harmonic maps between spheres $U : S^d \to S^d$. As emphasized above, the curvature of the domain manifold is irrelevant in the formation of point singularities, hence all the above results concerning the asymptotic dynamics of blow-up (in particular conjecture 1) remain valid in the case of a spherical domain. What makes the spherical domain interesting is a pattern of multiple blow-ups for high-degree initial maps. This is illustrated in figure 8 showing three consecutive blow-ups at the north pole, south pole, and again the north pole (animated simulations can be found at [19]). At each blow-up the degree of the map changes by one and eventually the solution comes to rest at the zero energy constant map. Note that, in view of the monotonicity formula (8) and Struwe's theorem (asserting that for harmonic maps between compact manifolds the heat flow starting from an initial map with nonzero degree and sufficiently small energy must blow up in finite time),

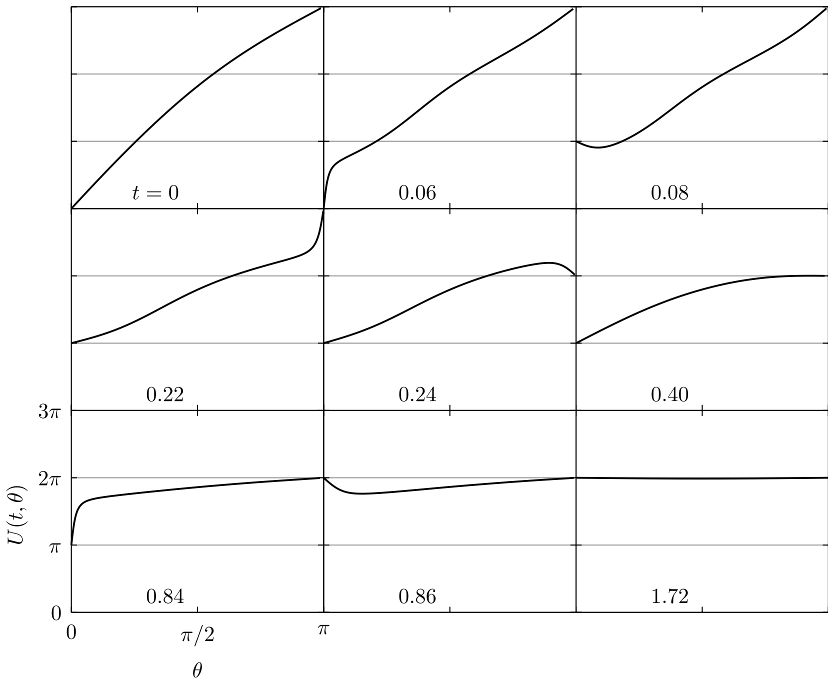

Figure 8. The solution of equation (4) starting from the initial map $U_0(\theta) = \sin\theta + 3\theta$ of degree 3. After three blow-ups the map becomes topologically trivial and settles down to the constant map.

conjecture 1 implies that the solution starting from an initial map of degree k must blow-up at least k times (note that the degree of the map need not decrease monotonically).

7. Final remarks

As mentioned in the introduction, the global weak peaking solutions (having the form of a shrinker and an expander glued together at infinity) exist for many supercritical heat flow equations, so it is natural to ask if these equations, similarly to the harmonic map flow, enjoy the uniqueness of continuation beyond the generic blow-up. We are currently investigating this question in the following models:

- *ℓ-equivariant harmonic map flow.* The corotational ansatz $(r, \phi) \to (u(r), \phi)$ is the special ($\ell = 1$) case of a more general ℓ-equivariant ansatz $(r, \phi) \to (u(r), \chi_\ell(\phi))$, where $\chi_\ell : S^{d-1} \to S^{d-1}$ is an eigenmap with constant energy density $k = \ell(\ell + d - 2)/2$. For ℓ-equivariant maps equation (9) changes to

$$u_t = \frac{1}{r^{d-1}} \left(r^{d-1} u_r\right)_r - \frac{k}{r^2} \sin(2u). \tag{72}$$

  All the qualitative results concerning existence of shrinkers and expanders and their linear stability obtained above for $\ell = 1$ trivially carry over to $\ell > 1$ provided that $3 \leqslant d < 2\ell + 2\sqrt{\ell} + 2$; however, the quantitative characteristics of self-similar solutions (in particular, those which imply the uniqueness of gluing an expander to the stable shrinker) remain to be checked.

- *Yang–Mills heat flow.* It is well known that there are close parallels between the harmonic map and the Yang–Mills heat flows [20]. For the spherically symmetric magnetic Yang–Mills potential $h(t, r)$ in $d \geqslant 3$ dimensions the analogue of equation (9) reads

$$h_t = \frac{1}{r^{d-3}} \left(r^{d-3} h_r\right)_r - \frac{d-2}{r^2} h(h-1)(h-2). \tag{73}$$

Using a similar shooting technique as in [3] one can easily show that for $5 \leqslant d \leqslant 9$ there are infinitely many shrinkers $h(t, r) = \phi_n(y)$. One novel feature, in comparison with the harmonic map flow, is that the first (stable) shrinker is known explicitly [21]:

$$\phi_1(y) = \frac{y^2}{b + ay^2}, \qquad b = \frac{1}{2}(6d - 12 - (d+2)\sqrt{2d-4}), \qquad a = \frac{\sqrt{d-2}}{2\sqrt{2}}. \tag{74}$$

This may be helpful in proving the Yang–Mills analogue of conjecture 1.
- *Semilinear heat equation.* The equation

$$u_t = \Delta u + |u|^{p-1} u, \tag{75}$$

for $d \geqslant 3$ and supercritical powers

$$\frac{d+2}{d-2} < p < p^* := \begin{cases} \infty & \text{for } 3 \leqslant d \leqslant 10, \\ 1 + \dfrac{6}{d-10} & \text{for } d \geqslant 11, \end{cases} \tag{76}$$

has self-similar solutions (shrinkers and expanders) [22–24] which give rise to global peaking solutions similar to the ones described in section 5; however, all these solutions are unstable [6] (cf also [25, 26]). It seems interesting to see if a kind of analogue of conjecture 1 holds for (codimension-one) threshold solutions.

In this paper we restricted our analysis to dimensions $3 \leqslant d \leqslant 6$. We wish to emphasize that this is not a technical restriction. For $d \geqslant 7$ the shrinkers disappear and consequently the blow-up changes character from type I to type II [27].

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