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Collapse of an instanton

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Abstract
We construct a two-parameter family of collapsing solutions to the 4 + 1 Yang–Mills equations and derive the dynamical law of the collapse. Our arguments indicate that this family of solutions is stable. The latter fact is also supported by numerical simulations.

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1. Introduction

Blow-up problems for nonlinear Schrödinger, wave and heat equations have been subjects of active research during the last 15 years (see [1–3] for reviews and [4, 5] for recent papers on the subject). A further surge of interest in blow-up for nonlinear wave equations has been recently motivated by their role in the understanding of the problem of singularity formation in general relativity (see [6] for a recent review). In this paper, we describe the asymptotic dynamics of blow-up for radial solutions of the semilinear wave equation:

$$\ddot{u} = \Delta u + \frac{1}{r^2} f(u), \tag{1.1}$$

in $\mathbb{R}^2$, where $u = u(t, r)$, r is the radial variable and

$$f(u) = 2u(1 - u^2). \tag{1.2}$$

Our analysis is applicable to a wider class of 'double-well' type of nonlinearities, $f(u)$, producing kink-type solutions, though some of these nonlinearities, e.g. wave map nonlinearities $f(u) = -\frac{1}{2}\sin(2u)$, lead to certain subtleties and will be considered elsewhere.

Before stating the results we show how equation (1.1) arises and put the problem in a broader context. We consider Yang–Mills (YM) fields in $(d + 1)$-dimensional Minkowski spacetime (in the following Latin and Greek indices take the values $1, 2, \ldots, d$ and $0, 1, 2, \ldots, d$, respectively). The gauge potential A_α is a one-form with values in the

Lie algebra g of a compact Lie group G. In terms of the curvature $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta]$ the YM equations take the form

$$\partial_\alpha F^{\alpha\beta} + [A_\alpha, F^{\alpha\beta}] = 0, \tag{1.3}$$

where [,] is the Lie bracket on G. For simplicity, we take here $G = SO(d)$ so the elements of $g = so(d)$ can be considered as skew-symmetric $d \times d$ matrices and the Lie bracket is the usual commutator. Assuming the spherically symmetric ansatz [7]

$$A_\mu^{ij}(x) = \left(\delta_\mu^i x^j - \delta_\mu^j x^i\right) \frac{1 - u(t, r)}{r^2}, \tag{1.4}$$

equations (1.3) reduce to the scalar semilinear wave equation for the magnetic potential $u(t, r)$:

$$\ddot{u} = \Delta_{(d-2)} u + \frac{d-2}{r^2} u(1 - u^2), \tag{1.5}$$

where $\Delta_{(d-2)} = \partial_r^2 + ((d-3)/r)\partial_r$ is the radial Laplacian in $d-2$ dimensions.

The central question for equation (1.5) is: can solutions starting from smooth initial data

$$u(0, r) = f(r), \qquad \dot{u}(0, r) = g(r) \tag{1.6}$$

become singular in the future? An answer to this question depends critically on the dimension d. To see why, we recall two basic facts. The first fact is the conservation of (positive definite) energy

$$E = \int_0^\infty \left(\dot{u}^2 + u'^2 + \frac{d-2}{2r^2}(1 - u^2)^2\right) r^{d-3}\, \mathrm{d}r. \tag{1.7}$$

The second fact is scale invariance of the YM equations: if $A_\alpha(x)$ is a solution of (1.3), so is $\tilde{A}_\alpha(x) = \lambda^{-1} A_\alpha(x/\lambda)$, or equivalently, if $u(t, r)$ is a solution of (1.5), so is $\tilde{u}(t, r) = u(t/\lambda, r/\lambda)$. Under this scaling, the energy scales as $\tilde{E} = \lambda^{d-4} E$, hence the YM equations are subcritical for $d \leqslant 3$, critical for $d = 4$, and supercritical for $d \geqslant 5$. In the subcritical case, shrinking of solutions to arbitrarily small scales costs an infinite amount of energy, so it is forbidden by energy conservation. This is a heuristic explanation of global regularity of the YM equations in physical dimensions, which was proved in [8, 9]. In contrast, in the supercritical case, shrinking of solutions might be energetically favourable and consequently singularities are anticipated. In fact, for $d \geqslant 5$, equation (1.5) admits self-similar solutions which are explicit examples of singularities [10, 11] and numerical simulations indicate that the stable self-similar solution determines the universal asymptotics of blow-up for large initial data [12].

In the critical dimension $d = 4$, the problem of singularity formation is more subtle because the scaling argument is inconclusive. In this case there are no smooth self-similar solutions; however, there is a family of static solutions $\chi(r/\lambda)$, where $\lambda > 0$ and

$$\chi(r) = \frac{1 - r^2}{1 + r^2}. \tag{1.8}$$

Using physicists' terminology we shall refer to this solution as the instanton. Numerical simulations indicate that the existence of the scale-free instanton plays a key role in the dynamics of blow-up, namely the blow-up has the universal profile of the instanton, whose size shrinks adiabatically to zero [12]. More precisely, it was conjectured in [12] that near the blow-up time t_* the solution has the form

$$u(t, r) \approx \chi\left(\frac{r}{\lambda(t)}\right), \tag{1.9}$$

where the scaling parameter $\lambda(t)$ tends to zero as $t \to t_*$. A natural question is: what determines the evolution of the scaling parameter; in particular, what is the asymptotic behaviour of $\lambda(t)$ for $t \to t_*$? In this paper, we address this question and show that for some initial conditions that are close to the instanton

$$\lambda(t) \sim \sqrt{\frac{2}{3}} \frac{t_* - t}{\sqrt{-\ln(t_* - t)}} \tag{1.10}$$

as $t \to t_*$. The logarithmic correction to the self-similar behaviour is characteristic for the blow-up in critical equations: it implies that the speed of blow-up goes asymptotically to zero and, consequently, no kinetic energy concentrates at the singularity (for a different approach see [13]).

2. Results

Thus, we consider the intial value problem (1.1), (1.2) and (1.6). Since we consider radial solutions only, the full Laplacian Δ can be replaced by the radial Laplacian $\Delta_r = (1/r)\partial_r r \partial_r$.

Our main result states that if the initial conditions (1.6) are sufficiently close to $(\chi(r/\lambda_0), -(\dot{\lambda}_0/\lambda_0)(r/\lambda_0)\chi'(r/\lambda_0))$, where $\lambda_0 > 0$ and $\dot{\lambda}_0 < 0$, then the resulting solution is of the form

$$u(r,t) = \chi\left(\frac{r}{\lambda}\right) + \mathrm{O}(\dot{\lambda}^2), \tag{2.1}$$

where the scaling parameter $\lambda = \lambda(t)$ satisfies the following equation:

$$\lambda\ddot{\lambda} = \tfrac{3}{4}\dot{\lambda}^4 \tag{2.2}$$

with the initial conditions λ_0 and $\dot{\lambda}_0$. In fact, our procedure allows us to find the solution $u(x,t)$ to any order in $\dot{\lambda}^2$ with the term of order $\dot{\lambda}^2$ given explicitly.

Note that solutions of equation (2.2) with the initial conditions such that $\dot{\lambda}_0 < 0$, decrease to zero as $t \to t_*$ for some t_* with $|\dot{\lambda}|$ decreasing so that our approximation improves as $t \to t_*$. This and equation (2.1) imply that the instanton collapses as $t \to t_*$.

To demonstrate the property of equation (2.2) mentioned above we note that equation (2.2) can be integrated explicitly. Indeed, we can rewrite (2.2) as $\dot{\lambda}^{-3}\ddot{\lambda} = \frac{3}{4}\lambda^{-1}\dot{\lambda}$ and integrate the resulting equation to obtain

$$-\tfrac{1}{2}\dot{\lambda}^{-2} = \tfrac{3}{4}(\ln\lambda + \ln c), \tag{2.3}$$

where $c > 0$. The latter equation can be rewritten as

$$\dot{\lambda}^2 \ln\left(\frac{1}{c\lambda}\right) = \frac{2}{3}. \tag{2.4}$$

This relation shows that we must have

$$c\lambda < 1.$$

Using equation (2.4) we obtain the equation for c in terms of λ_0:

$$\ln\left(\frac{1}{c\lambda_0}\right) = \frac{2}{3}\dot{\lambda}_0^{-2}. \tag{2.5}$$

We have two cases:

(a) $\dot{\lambda}_0 > 0$. Then, $c\lambda \uparrow 1$ and $\dot{\lambda} \uparrow \infty$ as t approaches some $t_* > 0$. Moreover, $c\lambda = 1 - (\frac{3}{2}c^2)^{1/3}(t_* - t)^{2/3}$ as $t \to t_*$.

(b) $\dot{\lambda}_0 < 0$. Then, $\dot{\lambda} < 0$ for $t > 0$ and there is $t_* > 0$ s.t. $\lambda \to 0$ as $t \uparrow t_*$. The value t_* can be found from (2.4):

$$t_* = \sqrt{\frac{3}{2}} \int_0^{\lambda_0} \mathrm{d}\lambda \ln^{1/2}\left(\frac{1}{c\lambda}\right). \tag{2.6}$$

Taking into account (2.5), this gives

$$t_* \approx \lambda_0 |\dot{\lambda}_0|^{-1}. \tag{2.7}$$

The time t_* is the point of collapse.

Note that in this case the function $|\dot{\lambda}|$ decreases as $t \to t_*$ as $[\ln(1/c\lambda)]^{-1/2}$ and, therefore, our approximation improves as $t \to t_*$.

Solutions of equation (2.2) with $\dot{\lambda}_0 < 0$ and $\lambda_0 > 0$ have the following asymptotics as $t \to t_*$

$$\lambda = \sqrt{\frac{2}{3}} \frac{t_* - t}{\sqrt{-\ln(t_* - t)}}. \tag{2.8}$$

In conclusion, we observe that equation (2.2) is invariant under the transformation

$$\lambda(t) \to \mu^{-1}\lambda(\mu t),$$

inherited from the invariance of the parent equation (1.1) under the scaling transformation

$$u(r,t) \to u(\mu r, \mu t).$$

3. Scaling transform and zero mode

A key role in our derivation is played by the fact that equation (1.1) is scale covariant under the transformation

$$u(r,t) \to u\left(\frac{r}{\lambda}, \frac{t}{\lambda}\right), \tag{3.1}$$

i.e. if $u(r,t)$ is a solution to (1.1), then so is $u(r/\lambda, t/\lambda)$. In particular, if $v(r)$ is a stationary solution, then so is $v(r/\lambda)$, $\lambda > 0$. The infinitesimal change of the instanton χ under this transformation is

$$\chi \to \chi + \delta\lambda\zeta, \tag{3.2}$$

where the function ζ is defined as

$$\zeta(r) := \partial_\lambda|_{\lambda=1}\chi\left(\frac{r}{\lambda}\right) = -r\partial_r\chi(r). \tag{3.3}$$

Explicitly,

$$\zeta(r) = \frac{4r^2}{(1+r^2)^2}. \tag{3.4}$$

Of course, ζ is the zero mode,

$$L\zeta = 0, \tag{3.5}$$

of the linearization of the rhs of (1.1) on χ, i.e. of the operator (recall, $\Delta_r = (1/r)\partial_r r\partial_r$)

$$L := -\Delta_r - \frac{1}{r^2} f'(\chi(r)). \tag{3.6}$$

(This operator is the variational or Fréchet derivative, $L = \partial\phi(\chi)$, of the map $\phi(u) = -\Delta u - (1/r^2)f(u)$ at the instanton χ.)

The following properties of the operator L will be important for us:

(a) $L = L^* \geqslant 0$;
(b) L has a simple eigenvalue at 0 with the eigenfunction ζ;
(c) the continuous spectrum of L fills $[0, \infty)$.

The first and third properties are obvious and the second property follows from the equation $L\zeta = 0$ and the fact that $\zeta > 0$ by the Perron–Frobenius theory (see [14, 15]).

Consider solutions of equation (1.1) of the form $u(r, t) := v(r/\lambda, t)$, where $\lambda > 0$ depends on t. Plugging the function $u(r, t) = v(r/\lambda, t)$ into equation (1.1), we obtain the following equation for v and λ:

$$\Delta_y v + y^{-2} f(v) = -\dot{\lambda}^2 B_1 v - \lambda\ddot{\lambda} B_2 v + \lambda^2 \partial_t^2 v - 2\lambda\dot{\lambda} B_2 \partial_t v, \tag{3.7}$$

where

$$B_1 = -y\partial_y - (y\partial_y)^2 \qquad \text{and} \qquad B_2 = y\partial_y. \tag{3.8}$$

4. Orthogonal decomposition

We look for a solution of equation (1.1) of the form

$$u(r, t) \equiv v\left(\frac{r}{\lambda}, t\right) = \chi\left(\frac{r}{\lambda}\right) + w\left(\frac{r}{\lambda}, t\right) \tag{4.1}$$

with $\lambda = \lambda(t)$ and

$$w(y, t) \qquad \text{small for all times.} \tag{4.2}$$

Moreover, to fix the splitting between the dynamics of λ and of w we require that w is orthogonal to the zero mode ζ:

$$\int_0^\infty \zeta(y) w(y, t) y \, \mathrm{d}y = 0. \tag{4.3}$$

The last two conditions will give us the dynamic law for λ.

Now we plug the decomposition $v(y, t) = \chi(y) + w(y, t)$ into equation (3.7) and use the fact that the function χ satisfies the equation

$$\Delta_y \chi + y^{-2} f(\chi) = 0, \tag{4.4}$$

to obtain the equation for w:

$$(L + \lambda^2 \partial_t^2) w = F(w, \lambda), \tag{4.5}$$

where, recall, $L = L_\chi$ is the linearized operator around χ:

$$L := -\Delta_y - y^{-2} f'(\chi) \tag{4.6}$$

and

$$F(w, \lambda) := \dot{\lambda}^2 B_1(\chi + w) + \ddot{\lambda}\lambda B_2(\chi + w) + y^{-2} N(w) + 2\lambda\dot{\lambda} B_2 \partial_t w \tag{4.7}$$

with the nonlinearity $N(w)$ defined by

$$N(w) := f(\chi + w) - f(\chi) - f'(\chi) w, \tag{4.8}$$

which in the case $f(u) = 2(1 - u^2)u$ gives

$$N(w) = -6\chi w^2 - 2w^3. \tag{4.9}$$

5. Perturbative analysis. Outline

We explain the main idea of our approach by proceeding formally and ignoring infrared divergences that arise. In the next section, we present a consistent perturbation theory. We look for a solution to equation (4.5) in the form

$$w(y,t) = \sum_{j\geqslant 1} \dot{\lambda}^{2j}\xi_j(y). \tag{5.1}$$

Plugging this expansion into equation (4.5), we arrive at a series of equations

$$L\xi_j = F_j(\xi_0,\ldots,\xi_{j-1}), \tag{5.2}$$

$j \geqslant 1$, where $\xi_0 = \chi$.

We demonstrate our approach by analysing the cases $j = 1$ and 2 in detail. We begin with $j = 1$. It is clear from (4.5)–(4.7) that

$$F_1(\xi_0) = B_1\chi. \tag{5.3}$$

Thus, ξ_1 satisfies the equation

$$L\xi_1 = B_1\chi. \tag{5.4}$$

Since, as we show in appendix A,

$$\int \zeta B_1\chi = 0, \tag{5.5}$$

equation (5.4) has a solution. The general solution of this equation is

$$\xi_1(y) = \xi_{10}(y) + \alpha_1\zeta(y) \tag{5.6}$$

for any $\alpha_1 \in \mathbb{R}$. Here

$$\xi_{10}(y) = -\frac{y^4}{(1+y^2)^2} \tag{5.7}$$

Now plugging $w = \dot{\lambda}^2\xi_1 + \mathrm{O}(\dot{\lambda}^4)$ into (4.3), and using (5.5) and $B_2\chi = -\zeta$, we obtain

$$\int \zeta\left(\dot{\lambda}^4 B_1\xi_1 - \ddot{\lambda}\lambda\zeta - \dot{\lambda}^4 6y^{-2}\chi\xi_1^2\right) = \mathrm{O}(\dot{\lambda}^6) + \mathrm{O}(\lambda\ddot{\lambda}\dot{\lambda}^2). \tag{5.8}$$

It will be shown in appendix A that the coefficients in front of α_1 and α_1^2 (remember (5.6)) vanish:

$$\int \zeta B_1\zeta - 12\int \zeta y^{-2}\chi\xi_{10}\zeta = 0 \tag{5.9}$$

and

$$\int \zeta y^{-2}\chi\zeta^2 = 0. \tag{5.10}$$

Therefore, relation (5.8) becomes

$$\ddot{\lambda}\lambda - \gamma\dot{\lambda}^4 = \mathrm{O}(\dot{\lambda}^6), \tag{5.11}$$

where

$$\gamma = \int \zeta\left(B_1\xi_{10} - 6y^{-2}\chi\xi_{10}^2\right)\left(\int \zeta^2\right)^{-1}. \tag{5.12}$$

We show in appendix A that $\gamma = \frac{3}{4}$ which, in the leading order, brings us to equation (2.8):

$$\lambda\ddot{\lambda} = \tfrac{3}{4}\dot{\lambda}^4. \tag{5.13}$$

As (5.11) shows, this equation is valid modulo the correction $\mathrm{O}(\dot{\lambda}^6)$.

Now, we proceed to the second term in (5.1) and derive a correction to equation (5.13). Remember that ξ_2 is defined by (5.2) with $j = 2$. Keeping in mind equation (5.13) we have

$$F_2(\xi_0, \xi_1) = B_1\xi_1 + \gamma B_2\chi - 6y^{-2}\chi\xi_1^2. \tag{5.14}$$

By (5.8) we have that

$$\int \zeta F_2(\xi_0, \xi_1) = 0, \tag{5.15}$$

so that the equation $L\xi_2 = F_2(\xi_0, \xi_1)$ is solvable. Now plugging

$$w = \dot\lambda^2\xi_1 + \dot\lambda^4\xi_2 + \mathrm{O}(\dot\lambda^6) \tag{5.16}$$

into (4.3), we obtain the equation for λ:

$$\lambda\ddot\lambda - \gamma\dot\lambda^4 + \delta\dot\lambda^6 + \varepsilon\dot\lambda^2\lambda\ddot\lambda + \mathrm{O}(\dot\lambda^8) = 0 \tag{5.17}$$

with δ and ε given in terms of integrals of ξ_1 and ξ_2 which can be explicitly computed. Observe that equation (5.17) is equivalent to the equation

$$\lambda\ddot\lambda - \gamma\dot\lambda^4 + (\delta - \gamma\varepsilon)\dot\lambda^6 + \mathrm{O}(\dot\lambda^8) = 0. \tag{5.18}$$

We can continue in the same manner to find the equation for λ to an arbitrary order in $\dot\lambda^2$.

Though the perturbation theory outlined above leads (as we will see in the next section) to correct—in the leading order—equations for the dilation parameter λ, it is, in fact, inconsistent. The leading correction, ξ_1, does not vanish at infinity and consequently the resulting solution has infinite energy. Worse, higher order corrections grow at infinity. Moreover, orthogonality condition (4.3) is not applicable (and, as a result, the parameter α_1 in (5.6) cannot be determined). The reason for this inconsistency is that the term $-\lambda^2\partial_t^2 w$ cannot be treated as a perturbation at large distances. A correct perturbation theory taking into account the leading contribution of this term at infinity is presented in the next section.

6. Perturbative analysis

In this section, we justify the formal analysis of section 5. We look for a solution, w, of equation (4.5) of the form

$$w(y, t) = \sum_{j\geqslant 1} \dot\lambda^{2j}\xi_j(y)\varphi_j(\dot\lambda^4 y^2, t). \tag{6.1}$$

We fix the functions ξ_j and φ_j by requiring that (a) ξ_j and φ_j are of the order O(1), (b) the functions φ_j satisfy the relations

$$\varphi_j(z, t) = 1 \qquad \text{for } z \ll 1 \tag{6.2}$$

and

$$\varphi_j(z, t) = \varphi_j(z) + \mathrm{O}(\dot\lambda^2), \tag{6.3}$$

(c) the following equations are satisfied

$$(L + \lambda^2\partial_t^2)\big(\dot\lambda^{2j}\xi_j\varphi_j\big) = \dot\lambda^{2j}F_j, \tag{6.4}$$

where the functions F_j are O(1) and depend only on $\xi_0\varphi_0, \ldots, \xi_{j-1}\varphi_{j-1}$ with $\xi_0 = \chi$ and $\varphi_0 \equiv 1$:

$$F_j \equiv F_j(\xi_0\varphi_0, \ldots, \xi_{j-1}\varphi_{j-1}) \tag{6.5}$$

and (d) the functions $\xi_j(y)$ satisfy the equations

$$L\xi_j = F_j(\xi_0, \dots, \xi_{j-1}). \tag{6.6}$$

As will be shown below, these requirements will define the functions ξ_j and φ_j uniquely, at least in the leading order.

We demonstrate our approach by analysing the cases $j = 1$ and 2 in detail. We begin with $j = 1$. It is clear from (4.5) to (4.7) that to

$$F_1(\xi_0\varphi_0) = B_1\chi. \tag{6.7}$$

Thus, ξ_1 satisfies equation (5.4). Recall that, due to (5.5), the latter equation is solvable and its general solution is given by (5.6). The constant α_1 in (5.6) is determined from the condition

$$\int \zeta \cdot \xi_1\varphi_1 = 0. \tag{6.8}$$

Since it plays no role in what follows we do not compute it here (see, however, (6.14) below).

Now, plugging $w = \dot\lambda^2\xi_1\varphi_1 + \mathrm{O}(\dot\lambda^4)$ into (4.3), omitting φ_1 (justification for this will be provided later) and using (5.5) and $B_2\chi = -\zeta$, we obtain (5.8), which, as shown in section 5, leads to (5.11) with $\gamma = \frac{3}{4}$.

Now we return to expansion (6.1) and find the equation for φ_1. Recall that φ_1 is defined through equations (6.2)–(6.6) with $j = 1$. We derive from these equations the equation for $\varphi_1(z, t)$ in the leading order in $\dot\lambda^2$ and in the domain $y \gg 1$. To this end we use equation (5.13) to estimate the order of higher derivatives of λ. In the leading order we can ignore the dependence of $\varphi_1(z, t)$ on t. Using equation (6.4) with $j = 1$ and equation (6.7) and using the fact that, for $y \gg 1$,

$$\xi_1 = -1 + \mathrm{O}\left(\frac{1}{y^2}\right), \qquad y\partial_y\xi_1 = -\frac{4}{y^2} + \mathrm{O}\left(\frac{1}{y^4}\right), \qquad B_1\chi = -\frac{4}{y^2} + \mathrm{O}\left(\frac{1}{y^4}\right), \tag{6.9}$$

we obtain after lengthy but elementary computations that φ_1 satisfies the equation

$$z^2\partial_z^2\varphi_1 + \left(z + \gamma z^2\right)\partial_z\varphi_1 - \left(1 - \tfrac{1}{2}\gamma z\right)\varphi_1 = -1. \tag{6.10}$$

To this we add the boundary conditions

$$\varphi_1(0) = 1. \tag{6.11}$$

The second boundary condition, $\varphi_1'(0)$, or, alternatively, an arbitrary constant in the general solution to (6.10)–(6.11), is found by matching the solution to (6.10), (6.11) in the region $z \ll 1$ with solution to (6.4) and (6.7) in the region $y \gg 1$. This is done in appendix B, where it is also shown that

$$\varphi_1(z) = \begin{cases} 1 - \dfrac{\gamma z}{4}\left(\ln\dfrac{z}{\dot\lambda^4} - \dfrac{7}{3}\right) & \text{for } z \ll 1 \\[2ex] \dfrac{c}{\sqrt{z}} + \dfrac{2}{\gamma z} + \dfrac{\bar c}{\sqrt{z}}\mathrm{e}^{-\gamma z} & \text{for } z \gg 1 \end{cases} \tag{6.12}$$

for some constants c and $\bar c$. Thus, we have

$$\dot\lambda^2\xi_1(y)\varphi_1(\dot\lambda^4 y^2) = \mathrm{O}(y^{-1}) \qquad \text{for } \dot\lambda^2 y \gg 1. \tag{6.13}$$

This implies, in particular, that the integral in (6.8) converges and it gives

$$\alpha_1 = \mathrm{O}\left(\ln\frac{1}{\dot\lambda^2}\right). \tag{6.14}$$

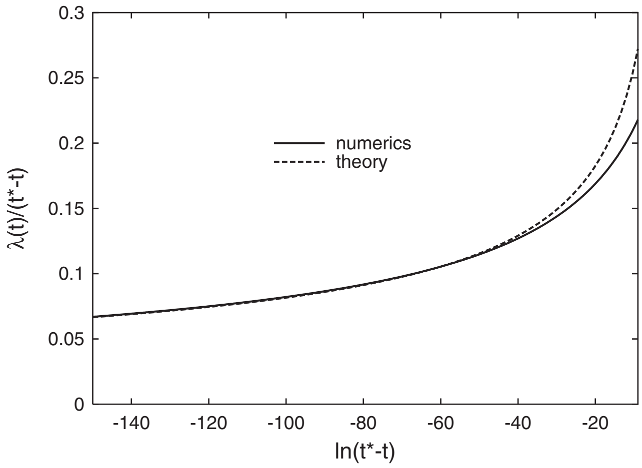

Figure 1. Dependence of the contraction parameter $\lambda(t)$ on time near the time of collapse t^*.

Now, we proceed to the second term in (6.1) and derive correction (5.17) to equation (5.11) (or (5.13)). Remember that ξ_2 is defined by (6.6) with $j = 2$. Keeping in mind equation (5.13) we choose

$$F_2(\xi_0\varphi_0, \xi_1\varphi_1) = B_1(\xi_1\varphi_1) + \gamma B_2\chi - 6y^{-2}\chi(\xi_1\varphi_1)^2. \tag{6.15}$$

From (5.8) we have that

$$\int \zeta F_2(\xi_0, \xi_1) = 0, \tag{6.16}$$

so that the equation $L\xi_2 = F_2(\xi_0, \xi_1)$ is solvable. Equations (6.2)–(6.6) with $j = 2$ imply an equation for φ_2 which is analysed in a similar way as the equation for φ_1. Now, plugging

$$w = \dot{\lambda}^2\xi_1\varphi_1 + \dot{\lambda}^4\xi_2\varphi_2 + O(\dot{\lambda}^6) \tag{6.17}$$

into (4.3) and setting φ_1 and φ_2 to 1, we obtain equation (5.17) (or (5.18)) for λ. We can continue in the same manner to find the equation for λ to an arbitrary order in $\dot{\lambda}^2$.

7. Conclusion

We found, perturbatively, a two-parameter family of collapsing solutions (parametrized by λ_0 and $\dot{\lambda}_0$) to the nonlinear wave equation (1.1)–(1.2) arising from the Yang–Mills equation in 4 + 1 dimensions. We also found the corresponding dynamics of collapse. The perturbation theory developed suggests that this family is (asymptotically) stable. This conclusion is supported by the numerical simulations we performed (some of the results of these simulations are given in figure 1).

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Appendix A. Computation of various integrals

In this appendix, we prove (5.5), (5.9), (5.10) and show that $\gamma = \frac{3}{4}$ (see equation (5.17)).

(1) (5.5). Recall $\zeta = -y\partial_y\chi(y)$ and $B_1 = -y\partial_y - (y\partial_y)^2$. Hence, $B_1\chi = (1 + y\partial_y)\zeta$ and

$$\int \zeta B_1\chi = \int \zeta(\zeta + y\zeta') = \int_0^\infty y\zeta(y\zeta)'\,\mathrm{d}y.$$

Integrating by parts we get

$$\int_0^\infty y\zeta(y\zeta)'\,\mathrm{d}y = \frac{1}{2}(y\zeta)^2\Big|_0^\infty$$

and since the boundary term vanishes, we get $\int \zeta B_1\chi = 0$.

In what follows we use the following relation ($m \leqslant n - 2$)

$$\int_0^\infty \frac{x^m \mathrm{d}x}{(1+x)^n} = \frac{m(m-1)\cdots 1}{(n-1)(n-2)\cdots(n-m-1)}.$$

(2) (5.9). Show that

$$\int \zeta B_1\zeta - 12\int \zeta y^{-2}\chi\xi_{10}\zeta = 0.$$

Compute

$$\begin{aligned}\int \zeta B_1\zeta &= -\int_0^\infty (\zeta\zeta' y^2 + \zeta\partial_y(y\zeta')y^2\,\mathrm{d}y \\ &= \int_0^\infty (\zeta\zeta' y^2 + \zeta'^2 y^3)\,\mathrm{d}y \\ &= \int_0^\infty (-\zeta^2 y + \zeta'^2 y^3)\,\mathrm{d}y.\end{aligned}$$

This gives

$$\begin{aligned}\frac{1}{8}\int \zeta B_1\zeta &= 2\int_0^\infty \left(\frac{-y^4}{(1+y^2)^4} + \frac{4y^4(1-y^2)^2}{(1+y^2)^6}\right) y\,\mathrm{d}y \\ &= \int_0^\infty \left(\frac{-x^2}{(1+x)^4} + \frac{4x^2(1-x)^2}{(1+x)^6}\right)\mathrm{d}x \\ &= -\frac{2\cdot 1}{3\cdot 2\cdot 1} + \frac{4\cdot 2\cdot 1}{5\cdot 4\cdot 3} - \frac{8\cdot 3\cdot 2\cdot 1}{5\cdot 4\cdot 3\cdot 2} + \frac{4\cdot 4\cdot 3\cdot 2\cdot 1}{5\cdot 4\cdot 3\cdot 2\cdot 1} = \frac{1}{5},\end{aligned}$$

$$\begin{aligned}\frac{1}{8}\int \zeta^2 y^{-2}\chi\xi_{10} &= -2\int_0^\infty \frac{y^4\cdot y^{-2}}{(1+y^2)^4}\,\frac{1-y^2}{1+y^2}\,\frac{y^4}{(1+y^2)^2}\,y\,\mathrm{d}y \\ &= -\int_0^\infty \frac{x^3(1-x)}{(1+x)^7}\,\mathrm{d}x \\ &= -\frac{3\cdot 2\cdot 1}{6\cdot 5\cdot 4\cdot 3} + \frac{4\cdot 3\cdot 2\cdot 1}{6\cdot 5\cdot 4\cdot 3\cdot 2} = \frac{1}{60}.\end{aligned}$$

Hence

$$\int \zeta B_1\zeta - 12\int \zeta y^{-2}\chi\xi_{10}\zeta = 8\left(\frac{1}{5} - \frac{12}{60}\right) = 0. \qquad \blacksquare$$

(3) (5.10). Compute

$$\frac{2}{4^3}\int \zeta^3 y^{-2}\chi = 2\int_0^\infty \frac{y^4}{(1+y^2)^6}\frac{1-y^2}{1+y^2}\,y\,\mathrm{d}y$$
$$= \int_0^\infty \frac{x^2(1-x)}{(1+x)^7}\,\mathrm{d}x$$
$$= \frac{2\cdot 1}{6\cdot 5\cdot 4} - \frac{3\cdot 2\cdot 1}{6\cdot 5\cdot 4\cdot 3} = 0.$$

(4) $\gamma = \frac{3}{4}$. Compute

$$\frac{1}{8}\int \zeta^2 = 2\int_0^\infty \frac{y^4}{(1+y^2)^4}\,y\,\mathrm{d}y$$
$$= \int_0^\infty \frac{x^2\mathrm{d}x}{(1+x)^4}$$
$$= \frac{2\cdot 1}{3\cdot 2\cdot 1} = \frac{1}{3}.$$

Compute

$$\int \zeta B_1 w_1 - 6\int \zeta y^{-2}\chi w_1^2 = 2,$$

so $\gamma = \frac{3}{4}$. ■

Appendix B. Solution φ_1

In this appendix, we find the solution to the 'initial' value problem (6.10)–(6.11) matching the solution to (5.4) with $j = 1$ in the region $1 \ll y \ll \dot{\lambda}^{-2}$. In the region $\{z \ll 1\}$ equations (6.10)–(6.11) have the general solution

$$\varphi_1 = 1 - \frac{\gamma}{4}\ln z(z+\cdots) + c'\frac{\gamma}{4}z(1+\cdots) \tag{B.1}$$

with an arbitrary constant c'.

For $z \gg 1$ equation (6.10) has the general solution

$$\varphi_1 = \frac{2}{\gamma z} + c\frac{1}{\sqrt{z}} + \bar{c}\frac{\mathrm{e}^{-\gamma z}}{\sqrt{z}}, \tag{B.2}$$

with arbitrary constants c and $\bar{c}$. Here, $1/\sqrt{z}$ and $\mathrm{e}^{-\gamma z}/\sqrt{z}$ are solutions of the corresponding homogeneous equation in the region $z \gg 1$.

It remains to find the constant c' in (B.1). To this end we match $\varphi_1(\dot{\lambda}^4 y^2)$ (in the leading order) to the solution of the equation

$$\left(L + \lambda^2\frac{\partial^2}{\partial t^2}\right) w = \dot{\lambda}^2 B_1\chi, \tag{B.3}$$

in the region $1 \ll y \ll \dot{\lambda}^{-2}$. We find the solution of the latter in the leading order in $\dot{\lambda}^2$ by a pertubation theory:

$$w = \dot{\lambda}^2 w_1 + \dot{\lambda}^6 w_2 + \cdots, \tag{B.4}$$

where $w_1 = \xi_1$ (see equation (5.6)). This implies the equation for w_2:

$$Lw_2 = 2\gamma\xi_{10}. \tag{B.5}$$

Two solutions of the corresponding homogeneous equation are (see (3.5))

$$\eta_1 = \frac{y^2}{(1+y^2)^2} \quad \text{and} \quad \eta_2 = \frac{y^2}{4} + \frac{3}{2} - \frac{13}{4(y^2+1)} - \frac{1}{4y^2(y^2+1)} + \frac{3y^2 \ln y^2}{(y^2+1)^2}. \tag{B.6}$$

By the method of variation of constants we obtain

$$w_2 = c_1\eta_1 + c_2\eta_2 \tag{B.7}$$

where the functions c_1 and c_2 are given by

$$c_1 = -\gamma \left\{ \frac{y^4}{8} + y^2 - \frac{4}{y^2+1} + \frac{1}{(y^2+1)^2} - \frac{y^6 \ln y^2}{(y^2+1)^3} - \frac{3y^4 \ln y^2}{2(y^2+1)^2} - \frac{3y^2 \ln y^2}{y^2+1} + 3\int_0^{y^2} \frac{\mathrm{d}s \ln s}{s+1} + 4 \right\} \tag{B.8}$$

and

$$c_2 = \gamma \left\{ \ln(y^2+1) + \frac{3}{y^2+1} - \frac{3}{2(y^2+1)} + \frac{1}{3(y^2+1)^3} - \frac{11}{6} \right\}. \tag{B.9}$$

Equations (B.7)–(B.9) for $y \gg 1$ yield

$$\begin{aligned} w_2 &= -\frac{\gamma}{8}y^2 + \gamma\frac{y^2}{4}\ln(1+y^2) + \cdots \\ &= \frac{\gamma}{4}y^2 \ln y^2 + \cdots \end{aligned} \tag{B.10}$$

Since, on the other hand, $w_1 = -1 + \mathrm{O}(y^2)$, we find in $y \gg 1$ that

$$w = -\dot{\lambda}^2 \left[1 - \frac{\gamma}{4} z \ln z + \cdots \right], \tag{B.11}$$

where $z = \dot{\lambda}^4 y^2$. Comparing (B.11) with (B.1) we find

$$c' = \ln y^2 - \ln(\dot{\lambda}^4 y^2) = \ln \dot{\lambda}^{-4} \tag{B.12}$$

and, therefore,

$$\varphi_1(z) = 1 - \frac{\gamma}{4} z \ln \frac{z}{\dot{\lambda}^4} + \cdots \qquad \text{for } z \ll 1. \tag{B.13}$$

Comparison of the numerically computed scaling parameter divided by $t^* - t$ with the analytic formula $\lambda(t)/(t^* - t) = \sqrt{2/3}(-\ln(t^* - t))^{-1/2}$. Note that there are no free parameters to be fitted. We believe that the apparent discrepancy (which is of the order of 10% at $\ln(t^* - t) = -60$) can be accounted for by including higher order corrections to the formula (1.10).

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