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Bifurcation and fine structure phenomena in critical collapse of a self-gravitating σ-field

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Abstract
Building on previous work on the critical behaviour in gravitational collapse of the self-gravitating $SU(2)$ σ-field and using high precision numerical methods we uncover a fine structure hidden in a narrow window of parameter space. We argue that this numerical finding has a natural explanation within a dynamical system framework of critical collapse.

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1. Introduction

Over the past few years the Einstein–$SU(2)$–σ model has attracted a great deal of attention [1–7]. This model is interesting because its rich phenomenology is sensitive to the value of a dimensionless parameter η characterizing the model which leads to various bifurcation phenomena. The most interesting bifurcation was found by Lechner *et al* [4] who showed that the critical behaviour in gravitational collapse changes character from continuous to discrete self-similarity when the coupling constant η increases above a critical value η_c. This phenomenon was interpreted in terms of dynamical systems theory as the homoclinic loop bifurcation where the two critical solutions, continuously self-similar (CSS) and discretely self-similar (DSS), merge in phase space. Since the echoing period Δ of the DSS solution diverges as $\eta \to \eta_c$, the numerical analysis of this bifurcation is extremely difficult and for this reason some of the aspects of critical behaviour near the bifurcation point, in particular the black-hole mass scaling law, were left open in [4].

Below, using high precision numerical methods, we confirm the main findings of [4]. In addition, we find that just above the bifurcation point the marginally supercritical side of the transition between dispersion and black holes exhibits a fine structure which is due to the competition between two coexisting critical solutions, the DSS one and the CSS one. The description of this phenomenon and its interpretation is the main purpose of this paper.

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The rest of the paper is organized as follows. For convenience, in section 2 we first briefly repeat the basic setting of the model and then we summarize what is known about it. In section 3 we present numerical results and finally, in section 4, we interpret them.

2. The model

The spherically symmetric Einstein–$SU(2)$–σ system is parametrized by three functions: the metric coefficients $A(t,r)$, $\delta(t,r)$ and the σ-field $F(t,r)$, which satisfy the following system of equations (see [2]) for the derivation):

$$\Box_g F - \frac{\sin(2F)}{r^2} = 0, \qquad \Box_g = -e^{\delta}\partial_t(e^{\delta}A^{-1}\partial_t) + \frac{e^{\delta}}{r^2}\partial_r(r^2 e^{-\delta}A\,\partial_r), \tag{1}$$

$$\partial_t A = -2\eta\, r A(\partial_t F)(\partial_r F), \tag{2}$$

$$\partial_r \delta = -\eta\, r\left((\partial_r F)^2 + A^{-2}e^{2\delta}(\partial_t F)^2\right), \tag{3}$$

$$\partial_r A = \frac{1-A}{r} - \eta\, r\left(A(\partial_r F)^2 + A^{-1}e^{2\delta}(\partial_t F)^2 + 2\,\frac{\sin^2 F}{r^2}\right), \tag{4}$$

where η is a dimensionless coupling constant. For $\eta = 0$ this system reduces to the σ model in Minkowski spacetime. The initial value problem for this system was studied by Bizoń *et al* [9] for $\eta = 0$ and by Husa *et al* [1] for $\eta > 0$. In these studies an important role is played by self-similar solutions. A countable family of continuously self-similar (CSS) solutions, hereafter denoted by CSS_n $(n = 0, 1, \ldots)$, was shown to exist for $0 \leqslant \eta < 0.5$ in [2, 5, 10]. These solutions are regular within the past light cone of the singularity, however they have a spacelike hypersurface of marginally trapped surfaces, i.e. an apparent horizon outside the past light cone if $\eta > \eta_n$, where η_n is an increasing sequence ($\eta_0 = 0.08$, $\eta_1 = 0.152$, etc). Linear stability analysis shows that the 'ground state' CSS_0 is stable while the excitations CSS_n have exactly n unstable modes.

Besides the CSS solutions, the system (1)–(4) has also a discretely self-similar (DSS) solution for $\eta \geqslant \eta_c \approx 0.17$. This solution was constructed by Lechner [3] via a pseudospectral method following the lines of Gundlach [8].

Next, we summarize what is known about the critical behaviour in gravitational collapse in this model. The first numerical studies of this problem, reported in [1], focused on relatively large coupling constants $\eta > 0.2$. In this range a 'clean' type-II critical DSS behaviour was observed; however the attempts to resolve critical evolutions for lower values of η encountered numerical difficulties and for $0.18 < \eta < 0.2$ only an approximate DSS behaviour was observed. Furthermore the echoing period Δ was found to increase sharply as the coupling constant decreases from 0.5 to 0.18. The critical behaviour for smaller couplings $0.1 < \eta < 0.2$ was studied in [4] (still smaller couplings are less interesting because then the model admits naked singularities). In the range $0.1 < \eta < 0.14$ a 'clean' CSS critical behaviour was observed; thus it became clear that somewhere in the interval $0.14 < \eta < 0.2$ there must be a transition between CSS and DSS critical solutions. The detailed studies of this transition [4] led to a conjecture that there exists a critical value of the coupling constant $\eta_c \approx 0.17$ for which the system exhibits the homoclinic loop bifurcation, i.e. the CSS saddle merges with the DSS limit cycle in the phase space. These results left open the question which of the two solutions in the transition region acts as the critical solution at the threshold of black-hole formation. In particular, near the bifurcation point the black-hole mass scaling could not be properly resolved.

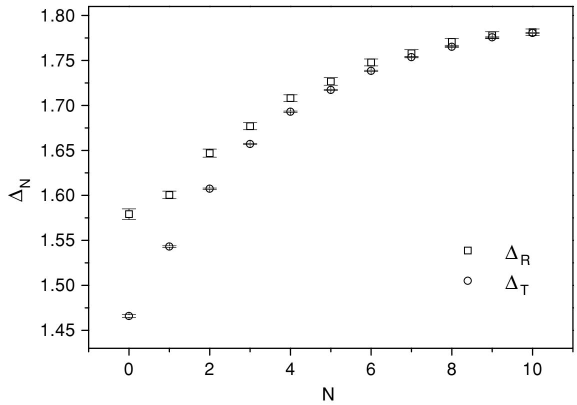


Figure 1. For $\eta = 0.1725$ we show Δ_r and Δ_t as functions of the cycle number N. Fitting the curve $\Delta + c\,\mathrm{e}^{-N}$ to the data we obtain $\Delta \approx 1.803$.

3. Numerical results

We have solved equations (1)–(4) for marginally critical initial data fine tuned to the DSS solution for coupling constants close to the critical value $\eta_c = 0.17$. Since the echoing period Δ increases sharply as the coupling constant tends to its critical value from above, it becomes more and more difficult to follow the evolution over a large number of DSS cycles[4]. We used the fully constrained implicit evolution scheme based on the second order accurate finite-difference Newton–Raphson scheme. In order to resolve the singular behaviour near the origin we used the grid which is uniform in $\ln(r)$. To get several cycles of the DSS attractor near the bifurcation point we had to fine tune parameters of initial data with a precision of 70 digits—this was achieved with the help of the Arprec Library [11]

Actually, it was not our aim to determine Δ with high precision, but rather to show that there exists a Δ to which the evolution converges. To this end, we determine Δ as a function of time (cycles) as the evolution approaches the limit cycle, i.e. the DSS solution. For a marginally critical solution we plot the function F versus $\ln(r)$ for some late time t_1 and superimpose the profile of the first echo at time t_2 shifted by $\ln(r) \to \ln(r) + \Delta_r$. The time t_2 and the radial echoing period Δ_r are chosen to minimize the discrepancy between two profiles. We also define the temporal echoing period Δ_t by the formula $t_2 = t^*(1 - \mathrm{e}^{-\Delta_t}) + \mathrm{e}^{-\Delta_t} t_1$, where t^* is the accumulation time. Repeating this calculation for a sequence of pairs (t_n, t_{n+1}), we get a sequence of values Δ_r and Δ_t. Of course, if the evolution converges to the DSS solution, both Δ_r and Δ_t should converge to the same constant. In figure 1 we show the convergence of Δ_r and Δ_t during a critical evolution for the coupling constant $\eta = 0.1725$. Note that the curve levels off, thus signalling the closeness of the evolution to the limit cycle. As the coupling constant decreases, Δ grows (see figure 2) and the approach to the DSS solution becomes slower.

Let p^* be a critical parameter value which separates dispersion from black holes (this value can be found by standard bisection). In agreement with [1] we find that for $\eta > 0.17$ the solution corresponding to p^* is DSS, in particular for p slightly below p^* we observe DSS subcriticality, i.e. the solution approaches the DSS solution and then disperses. Looking at the

[4] By an elementary dimensional analysis the number of cycles scales as $N \sim -(1/\lambda\Delta) \ln|p - p^*|$, where Δ is the echoing period and λ is the eigenvalue of the growing mode of the DSS solution.

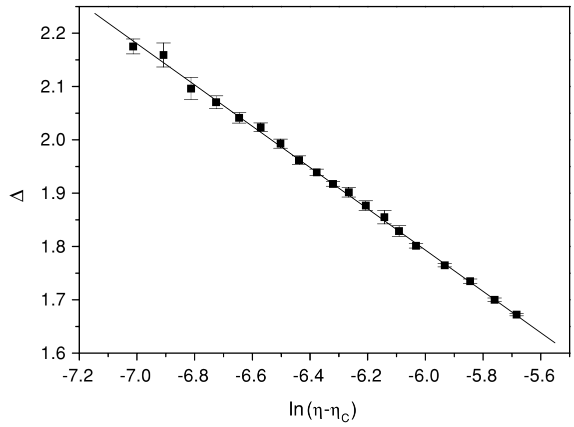


Figure 2. Fitting the echoing period Δ (determined as in figure 1) to the analytic prediction $\Delta = a \ln|\eta - \eta_c| + \text{const.}$ we get $\eta_c = 0.1701$ and the slope $a = -0.389$ which is in very good agreement with the analytic prediction $a = -2/\lambda_{\mathrm{CSS}}$ and the linear perturbation result [3] $\lambda_{\mathrm{CSS}}(\eta_c) \approx 5.14$.

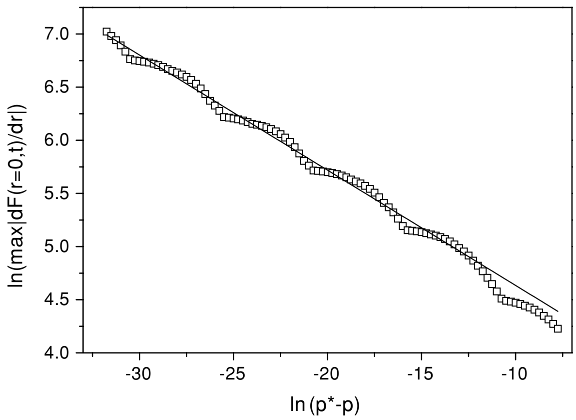


Figure 3. The subcritical scaling (5) for the coupling constant $\eta = 0.19$. The slope of the linear fit is approximately equal to -0.109. The wiggles, which are imprints of discrete self-similarity, have the period $\approx$4.8.

maximum value of the spatial derivative of the scalar field at the origin as a function of p, we find a typical subcritical scaling law (see figure 3)

$$\max|\partial_r F(t,0)| \sim (p^* - p)^{-\gamma_{\mathrm{DSS}}}. \tag{5}$$

For $p > p^*$ black holes are formed, however this happens in a rather unusual manner. This is shown in figure 4 where the metric function A is seen to develop two minima very close to zero which signals an almost simultaneous formation of a small and a large marginally trapped surface.

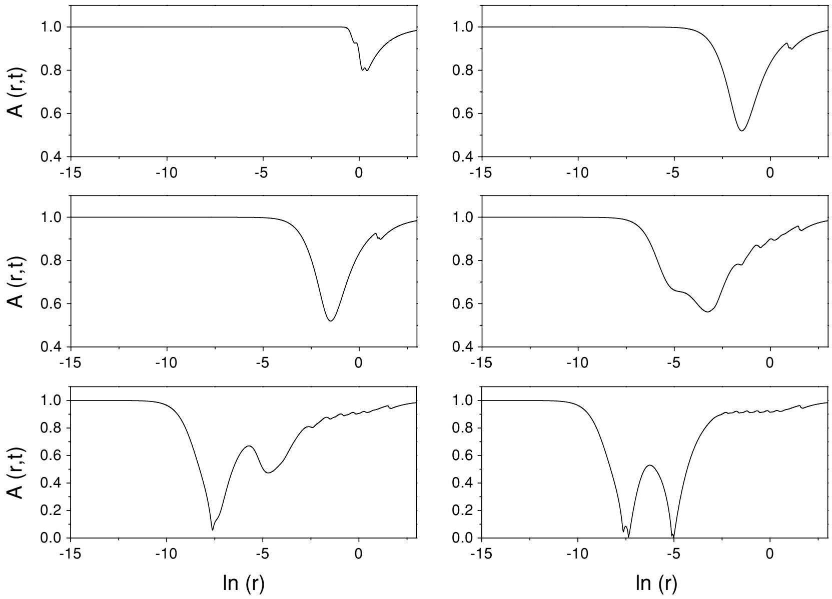


Figure 4. The series of snapshots of the metric function $A(t, r)$ from the evolution of marginally supercritical initial data for the coupling constant $\eta = 0.19$. In the last frame one can see the formation of two apparent horizons.

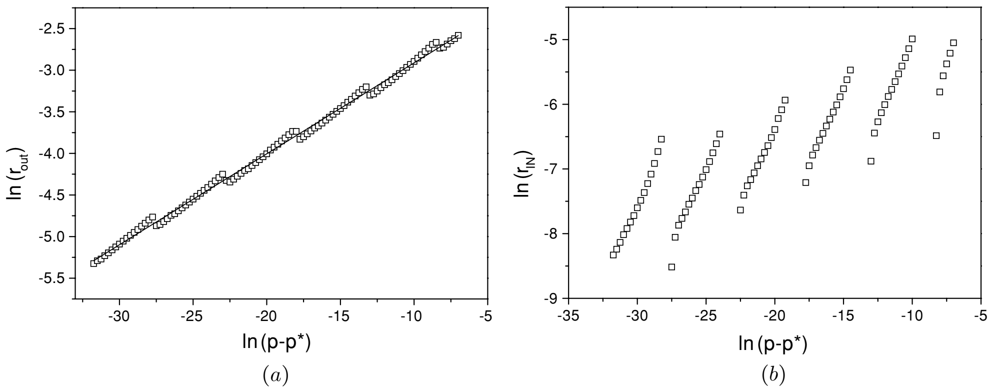


Figure 5. $\eta = 0.19$. (*a*) The locus of the outer apparent horizon is shown to satisfy the power law (6) with the slope $\gamma_{\mathrm{DSS}} = 0.109$. The wiggles superimposed on the linear fit have the period 4.8 which agrees with the analytic prediction $\Delta/2\gamma_{\mathrm{DSS}}$. (*b*) The locus of the inner horizon does not follow a power law. The jump discontinuities are periodic with the period 4.8.

Let us denote their apparent horizon radii by r_{in} and r_{out}, respectively. We find that the outer radius exhibits the standard DSS supercritical scaling (see figure 5(*a*))

$$r_{\mathrm{out}} \sim (p - p^*)^{\gamma_{\mathrm{DSS}}}, \tag{6}$$

but the inner radius does not seem to scale. The latter fact was already mentioned in [4]. The corresponding graph shows a see-saw structure, i.e. short straight lines with jump discontinuities at certain values of the parameter p (see figure 5(*b*)).

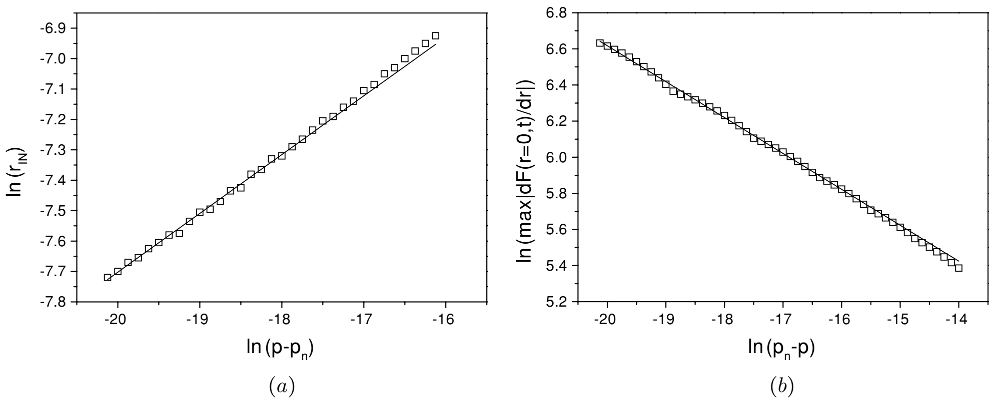


Figure 6. $\eta = 0.19$. (*a*) The supercritical (8) and (*b*) subcritical (9) scalings around p_n for $n = 2$ (we get the same picture for each n). The slopes of the linear fits are equal to ± 0.195 which agrees with the analytic prediction $\pm 1/\lambda_{\mathrm{CSS}}$ where $\lambda_{\mathrm{CSS}}(\eta = 0.19) = 5.1$ was obtained independently from the linear perturbation theory by Lechner [3].

Table 1. The first five critical parameter values p_n for the coupling constant $\eta = 0.19$.

n	1	2	3	4	5	∞
p_n	0.529 001 923 689	0.528 771 570 563	0.528 769 577 618	0.528 769 560 376	0.528 769 560 227	0.528 769 560 226

In order to understand this strange behaviour we looked in more detail at the evolution of initial data fine tuned to the location of these jumps. With the help of arbitrary precision numerical methods we found the following remarkable structure: for a given family of initial data there is a sequence of discrete parameter values $p_1 > p_2 > p_3 \cdots > p_n$ such that a solution with $p \in (p_n, p_{n+1})$ approaches the CSS_1 solution n times, i.e. the solution comes close to the CSS_1 solution, turns away and returns n times before leading to black-hole formation. Multiple approaches to the CSS_1 solution were already noted in [4] where they were called episodic CSS; however the corresponding fine structure in the parameter space was not seen there. The sequence $\{p_n\}$ with $n \leqslant 5$ is listed in table 1.

Because of numerical limitations we were not able to resolve higher p_n, however the data shown in table 1 seem to indicate that the sequence p_n converges to p^* as n tends to infinity. Actually, we find that the two consecutive parameters p_n satisfy the scaling law

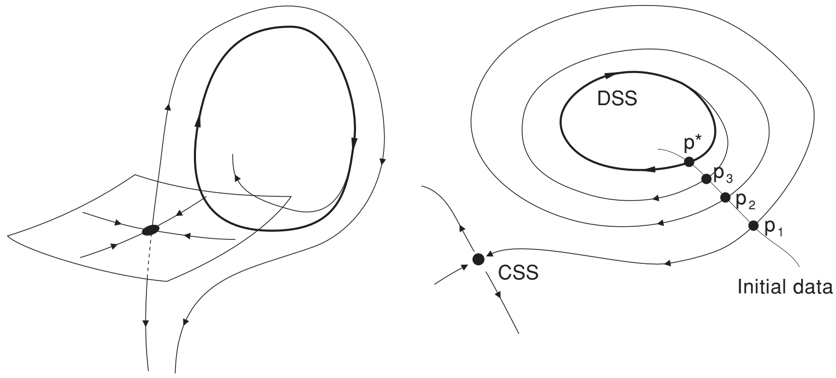
$$\frac{p_n - p^*}{p_{n+1} - p^*} \approx \exp\left(\frac{\Delta}{2\gamma_{\mathrm{DSS}}}\right). \tag{7}$$

Now we return to the problem of scaling of the inner horizon radius r_{in}. For $p = p_n + \varepsilon$, i.e. for p just above one of the p_n we see a clear CSS scaling (see figure 6(*a*))

$$r_{\mathrm{in}} \sim (p - p_n)^{\gamma_{\mathrm{CSS}}}. \tag{8}$$

For $p = p_n - \varepsilon$ the solution displays a kind of pseudo-dispersion after its last CSS episode. This pseudo-dispersion manifests itself as follows: after leaving the CSS solution, the maximum of the function F decreases, the inner minimum of A disappears and later a spike develops which leads to the formation of an apparent horizon at r_{out}. In this range of p we see the subcritical CSS scaling (see figure 6(*b*))

$$\max|\partial_r F(t, 0)| \sim (p_n - p)^{-\gamma_{\mathrm{CSS}}}, \tag{9}$$



(*a*) (*b*)

Figure 7. (*a*) Shil'nikov bifurcation. (*b*) The conjectured phase space picture.

however the masses of black holes formed in such evolutions are 'large' and do not scale. We remark that since the solutions on both sides of p_n form black holes, the bisection which gives critical parameter values p_n has to be performed in a sense 'by hand'.

4. Interpretation of numerical results

The results presented above confirm and extend the findings of Lechner *et al* [4]. Probing the bifurcation point η_c with higher accuracy we improved the evidence that Δ diverges as η tends to the critical value $\eta_c = 0.17$ from above, which in turn confirms the picture that the DSS cycle merges with the CSS solution at the critical coupling constant η_c. A natural question is: what is the meaning of the series of critical parameter values p_n within this picture?

We conjecture that our system shows a so-called Shil'nikov bifurcation [12]. In his classification of loop bifurcations for three-dimensional systems, Shil'nikov considered a system with a saddle point together with a homoclinic orbit which bifurcates for some value of a parameter. Assuming that the eigenvalues of the saddle point are real and satisfy the following conditions: $\lambda_1 > 0 > \lambda_2 > \lambda_3$ and $\lambda_1 + \lambda_2 > 0$ (plus some less important technical conditions), Shil'nikov showed that a saddle limit cycle bifurcates and the phase space picture looks qualitatively as in figure 7(*a*).

Of course, our system is infinite dimensional and the Shil'nikov theorem cannot be applied directly. Nevertheless, it is expected that a similar picture to figure 7(*a*) will be valid for higher-dimensional systems as long as only a few largest eigenvalues of the perturbation matter. Recently, Donninger [13] has studied linear perturbations around the CSS solution and found that for coupling constants around the critical value η_c the first three largest eigenvalues do in fact satisfy the above-stated Shil'nikov conditions. Combining this property with the fact that the bifurcating DSS solution is a saddle limit cycle, we conjecture that the (one-dimensional) unstable manifold of the DSS solution lies on the stable manifold of the neighbouring CSS solution. This is sketched in figure 7(*b*). More precisely, the DSS unstable manifold winds around the limit cycle (infinitely many times) and eventually runs into the CSS saddle. Suppose that a curve of initial data intersects this spiral manifold at values p_n, with $\lim p_n = p^*$, where p^* corresponds to the intersection with the limit cycle. Then, the dynamical behaviour will have exactly the form we observed above: for p equal to one of the p_n, the solution spirals n times around the limit cycle each time coming closer to the CSS saddle before hitting it. For $p = p_n \pm \varepsilon$, the behaviour is similar, except that the solution does not hit the CSS saddle but escapes along its unstable manifold. This is the reason why one observes the CSS scaling around p_n with an exponent related to the unstable eigenvalue of the CSS solution. Note that the scaling law (7) follows immediately from the picture shown in figure 7(*b*) because during one cycle of evolution the distance from the DSS limit cycle increases by the factor $e^{\lambda_{\mathrm{DSS}}\Delta}$ (and $\gamma_{\mathrm{DSS}} = 1/\lambda_{\mathrm{DSS}}$).

Acknowledgments

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