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Remark on the absence of self-similar solutions in a 4 + 1 vacuum gravitational collapse

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Abstract
We give a very short proof that the vacuum Einstein equations in 4+1 dimensions have no cohomogeneity-two Bianchi IX continuously self-similar solutions.

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1. Introduction

In a recent paper [1] it was shown that in five spacetime dimensions one can perform a consistent cohomogeneity-two symmetry reduction of the vacuum Einstein equations which—in contrast to the spherically symmetric reduction—admits time-dependent asymptotically flat solutions. The key idea was to modify the standard spherically symmetric ansatz by replacing the round metric on the three-sphere with the homogeneously squashed metric, thereby breaking the $SO(4)$ isometry to $SO(3) \times U(1)$. In this way the squashing parameter becomes a dynamical degree of freedom and Birkhoff's theorem is evaded. This model (which we shall refer to as the BCS model) provides a simple theoretical setting for studying the dynamics of gravitational collapse in vacuum. Numerical simulations indicate that the spherically symmetric solutions, Minkowski and Schwarzschild, play the role of attractors in the evolution of generic regular initial data (small and large ones, respectively) and the transition between these two outcomes of evolution exhibits a discretely self-similar critical behavior [1]. In this respect the BCS model is very similar to the Einstein-massless scalar field system [2–4]. However, there is one interesting difference between these two models which we want to point out here. The difference is concerned with the existence of continuously self-similar (CSS) solutions. In [5] Christodoulou proved that the Einstein-massless scalar field system possesses CSS solutions. These solutions, suitably truncated, provide examples of naked singularities developing from regular initial data (however, being unstable [6], they do not contradict the weak cosmic censorship conjecture). We will show below that the BCS model has no CSS solutions. This result indicates that the CSS naked singularities found by Christodoulou for the self-gravitating massless scalar field are, in a sense, matter generated (mathematically, they are related to the fact that only derivatives of the scalar field appear in the equations).

2. The BCS ansatz and self-similarity

After [1] we parametrize the metric as follows:

$$\mathrm{d}s^2 = -A\,\mathrm{e}^{-2\delta}\,\mathrm{d}t^2 + A^{-1}\,\mathrm{d}r^2 + \tfrac{1}{4}r^2\left(\mathrm{e}^{2B}\left(\sigma_1^2+\sigma_2^2\right)+\mathrm{e}^{-4B}\sigma_3^2\right), \tag{1}$$

where A, δ, and B are the functions of (t, r), and σ_i are left invariant one-forms on $SU(2)$ which in terms of the Euler angles take the form

$$\begin{aligned}\sigma_1 &= \cos\psi\ \mathrm{d}\theta + \sin\psi\ \sin\theta\ \mathrm{d}\phi,\\ \sigma_2 &= -\sin\psi\ \mathrm{d}\theta + \cos\psi\ \sin\theta\ \mathrm{d}\phi, \qquad \sigma_3 = \mathrm{d}\psi + \cos\theta\ \mathrm{d}\phi.\end{aligned} \tag{2}$$

Substituting this ansatz into the vacuum Einstein equations we get the following system of PDEs:

$$\partial_r A = -\frac{2A}{r} + \frac{2}{3r}(4\,\mathrm{e}^{-2B} - \mathrm{e}^{-8B}) - 2r(\mathrm{e}^{2\delta}A^{-1}(\partial_t B)^2 + A(\partial_r B)^2), \tag{3}$$

$$\partial_t A = -4rA\,(\partial_t B)\,(\partial_r B), \tag{4}$$

$$\partial_r \delta = -2r(\mathrm{e}^{2\delta}A^{-2}(\partial_t B)^2 + (\partial_r B)^2), \tag{5}$$

$$\partial_t(\mathrm{e}^{\delta}A^{-1}r^3\partial_t B) = \partial_r(\mathrm{e}^{-\delta}Ar^3\partial_r B) + \frac{4}{3}\mathrm{e}^{-\delta}r(\mathrm{e}^{-2B} - \mathrm{e}^{-8B}). \tag{6}$$

These equations have the scaling symmetry $(t, r) \to (\lambda t, \lambda r)$ so it is natural to look for continuously self-similar (CSS) solutions, that is, solutions which are scale invariant. Such solutions depend on a single variable $\rho = r/t$ and then the system (3)–(6) reduces to ordinary differential equations (where prime is $\mathrm{d}/\mathrm{d}\rho$ and $Z = \mathrm{e}^{\delta}\rho/A$)

$$\rho A' = -2A + \frac{2}{3}(4\,\mathrm{e}^{-2B} - \mathrm{e}^{-8B}) - 2\rho^2 A(1+Z^2)B'^2, \tag{7}$$

$$A' = -4\rho A B'^2, \tag{8}$$

$$\rho Z' = Z + 2Z(1-Z^2)\rho^2 B'^2, \tag{9}$$

$$B'' = \frac{(2Z^2-3)B' + 2\rho^2(1-Z^4)B'^3}{\rho(1-Z^2)} + \frac{4}{3}\frac{\mathrm{e}^{-2B} - \mathrm{e}^{-8B}}{\rho^2 A(1-Z^2)}. \tag{10}$$

The combination of equations (7) and (8) yields the constraint

$$3A - 3A(1-Z^2)\rho^2 B'^2 - 4\,\mathrm{e}^{-2B} + \mathrm{e}^{-8B} = 0. \tag{11}$$

We are interested in regular solutions, where 'regular' means twice continuously differentiable. At the origin regular solutions must satisfy the following initial conditions:

$$B(\rho) \sim b\rho^2, \qquad A(\rho) \sim 1 - 4b^2\rho^4, \qquad Z(\rho) \sim \rho, \tag{12}$$

where we used the remaining scaling freedom to set $Z'(0) = 1$ for convenience. It follows from (12) and equation (9) that if $Z < 1$, then $Z(\rho) \geqslant \rho$; hence, there is a ρ_0 such that $Z(\rho_0) = 1$. Geometrically, ρ_0 corresponds to the similarity horizon (the light cone of the singularity).

Proof. We will show that solutions starting from initial conditions (12) cannot be regular at ρ_0. Assume for contradiction that the solution $(A(\rho), Z(\rho), B(\rho))$ is regular on the closed interval $I = \{\rho : 0 \leqslant \rho \leqslant \rho_0\}$. First, note that the function A is positive on I since from equation (8) we have $A(\rho) = \exp\left(-4\int_0^\rho sB'(s)^2\,\mathrm{d}s\right)$. Second, it follows from equation (10) that if $B'(\rho_1) = 0$ for some ρ_1, then $B''(\rho_1)$ has the same sign as $B(\rho_1)$. Thus, the function $B(\rho)$ is monotone on I and $B'(\rho)$ has the sign of b. Next, let us define the function

$$H = 8\,\mathrm{e}^{-2B} - 5\,\mathrm{e}^{-8B} - 3\rho A B' - 3A. \tag{13}$$

With the use of this function (which we found by an arduous trial and error), the rest of the proof amounts to a one-line exercise in elementary calculus. The initial conditions (12) imply that $H(\rho) \sim 9b\rho^2$ near the origin. Differentiating H and using the constraint (11), we obtain

$$H' + \left(\frac{1}{\rho(1-Z^2)} + 3B'\right) H = 27\,\mathrm{e}^{-8B} B'. \tag{14}$$

Hence, $H(\rho)$ cannot have a zero for $\rho < \rho_0$ because if $H(\rho) = 0$, then $H'(\rho)$ has the same sign as $B'(\rho)$ and therefore b. Similarly, $H(\rho_0)$ cannot vanish because L'Hopital's rule gives $H'(\rho_0) = 54\,\mathrm{e}^{-2B(\rho_0)} B'(\rho_0)$. However, $H(\rho_0)$ must vanish for regular solutions, as follows immediately from (14). This contradiction ends the proof. □

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