Ground State of the Conformal Flow on $\mathbb{S}^3$

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Abstract

We consider the conformal flow model derived in Bizoń et al. (2017) as a normal form for the conformally invariant cubic wave equation on $\mathbb{S}^3$. We prove that the energy attains a global constrained maximum at a family of particular stationary solutions that we call the ground state family. Using this fact and spectral properties of the linearized flow (which are interesting in their own right due to a supersymmetric structure), we prove nonlinear orbital stability of the ground state family. The main difficulty in the proof is due to the degeneracy of the ground state family as a constrained maximizer of the energy. © 2018 Wiley Periodicals, Inc.

1 Introduction

Long-time behavior of nonlinear dispersive waves on a compact manifold can be very rich and complex because, in contrast to unbounded domains, waves cannot disperse to infinity and keep self-interacting for all times (even the global regularity of arbitrarily small solutions is a nontrivial issue). A major mathematical challenge in this context is to describe the energy transfer between eigenmodes of the corresponding linearized flow near the zero equilibrium. A simple model for gaining insight into this problem is the conformally invariant cubic wave equation on the three-sphere. The key feature of this model is the fully resonant linearized spectrum. As a consequence, the long-time behavior of small solutions of this equation is well approximated by solutions of an infinite-dimensional time-averaged Hamiltonian system that governs resonant interactions between the modes. This system, called the *conformal flow* on $\mathbb{S}^3$, has been introduced and studied in [3].

Among its many remarkable features (in particular, low-dimensional invariant subspaces), the conformal flow has been found to admit a wealth of *stationary states*, i.e., solutions for which no energy transfer between the modes occurs. In this paper we show that among the stationary states there is a distinguished one,

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hereafter called the *ground state*, which is a global constrained maximizer of the energy. The main body of the paper is devoted to proving nonlinear orbital stability of the ground state.

In terms of complex Fourier coefficients $(\alpha_n(t))_{n\in\mathbb{N}}$, the conformal flow system takes the form (see [3] for the details of the derivation)

$$i(n+1)\frac{d\alpha_n}{dt} = \sum_{j=0}^{\infty}\sum_{k=0}^{n+j} S_{njk,n+j-k}\bar{\alpha}_j\alpha_k\alpha_{n+j-k}, \tag{1.1}$$

where

$$S_{njk,n+j-k} = \min(n,j,k,n+j-k)+1.$$

This is the Hamiltonian system with the conserved energy function

$$H(\alpha) = \sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{n+j} S_{njk,n+j-k}\,\bar{\alpha}_n\bar{\alpha}_j\alpha_k\alpha_{n+j-k} \tag{1.2}$$

and symplectic form $\sum_n 2i(n+1)\,d\bar{\alpha}_n \wedge (-d\alpha_n)$ so that equations of motion (1.1) can be written in the form

$$i(n+1)\frac{d\alpha_n}{dt} = \frac{1}{2}\frac{\partial H}{\partial \bar{\alpha}_n}. \tag{1.3}$$

The conformal flow system enjoys the following three one-parameter groups of symmetries:

$$\text{Scaling: } \alpha_n(t) \to c\alpha_n(c^2 t), \tag{1.4}$$

$$\text{Global phase shift: } \alpha_n(t) \to e^{i\theta}\alpha_n(t), \tag{1.5}$$

$$\text{Local phase shift: } \alpha_n(t) \to e^{in\mu}\alpha_n(t), \tag{1.6}$$

where c, θ, and μ are real parameters. The latter two symmetries give rise to two additional conserved quantities:

$$Q(\alpha) = \sum_{n=0}^{\infty}(n+1)|\alpha_n|^2, \tag{1.7}$$

$$E(\alpha) = \sum_{n=0}^{\infty}(n+1)^2|\alpha_n|^2. \tag{1.8}$$

NOTATION: We denote the set of nonnegative integers by $\mathbb{N}$ and the set of positive integers by $\mathbb{N}_+$. A sequence $(\alpha_n)_{n\in\mathbb{N}}$ is denoted for short by α. The space of square-summable sequences is denoted by ℓ^2. Given $s > 0$, we define the weighted space of sequences

$$h^s := \left\{\alpha \in \ell^2(\mathbb{N}) \colon \sum_{n=0}^{\infty}(n+1)^{2s}|\alpha_n|^2 < \infty\right\} \tag{1.9}$$

endowed with its natural norm. We write $X \lesssim Y$ to denote the statement that $X \leq CY$ for some universal (i.e., independent of other parameters) constant $C > 0$.

The following theorem states that the conformal flow is globally well-posed in h^1. This is not an optimal result; however, it is sufficient for our purposes. Note that all three conserved quantities H, Q, and E are well-defined in h^1.

THEOREM 1.1. *For every initial data $\alpha(0) \in h^1$, there exists a unique global-in-time solution $\alpha(t) \in C(\mathbb{R}, h^1)$ of the system* (1.1). *Moreover, for every t,*

$$H(\alpha(t)) = H(\alpha(0)), \quad Q(\alpha(t)) = Q(\alpha(0)), \quad E(\alpha(t)) = E(\alpha(0)).$$

A solution of (1.1) is called a *stationary state* if $|\alpha(t)|$ is time-independent. A stationary state is called a *standing wave* if it has the form

$$\alpha(t) = Ae^{-i\lambda t}, \tag{1.10}$$

where the complex amplitudes $(A_n)_{n\in\mathbb{N}}$ are time-independent and the parameter λ is real. Substituting (1.10) into (1.1), we get a nonlinear system of algebraic equations for the amplitudes:

$$(n+1)\lambda A_n = \sum_{j=0}^{\infty} \sum_{k=0}^{n+j} S_{njk,n+j-k} \bar{A}_j A_k A_{n+j-k}. \tag{1.11}$$

This equation can be written as

$$\lambda \frac{\partial Q}{\partial \bar{A}_n} = \frac{1}{2} \frac{\partial H}{\partial \bar{A}_n};$$

hence standing waves (1.10) admit a variational characterization as the critical points of the functional

$$K(\alpha) = \frac{1}{2} H(\alpha) - \lambda Q(\alpha). \tag{1.12}$$

Equivalently, standing waves (1.10) are critical points of H for fixed Q, where λ is a Lagrange multiplier.

The simplest solutions of (1.11) are the *single-mode* states, which are given, for any $N \in \mathbb{N}$, by

$$A_n = c\delta_{nN}, \quad \lambda = |c|^2, \quad c \in \mathbb{C}. \tag{1.13}$$

In this paper we are concerned with the following family of standing waves that bifurcates from the $N = 0$ single-mode state:

$$A_n = cp^n, \quad \lambda = \frac{|c|^2}{(1-|p|^2)^2}, \tag{1.14}$$

where $c, p \in \mathbb{C}$ and $|p| < 1$. We shall refer to (1.14) as the *ground state* family because it is the global maximizer of H for fixed Q, as follows from our next theorem.

THEOREM 1.2. *For every $\alpha \in h^{1/2}$ the following inequality holds*:

$$H(\alpha) \leq Q(\alpha)^2. \tag{1.15}$$

Moreover, $H(\alpha) = Q(\alpha)^2$ *if and only if* $\alpha_n = cp^n$ *for some* $c, p \in \mathbb{C}$ *with* $|p| < 1$.

The ground state family (1.14) is parametrized by two complex parameters c and p with $|p| < 1$. In what follows, by the ground state we shall mean the normalized solution $A(p)$ for $\lambda = 1$ with

$$A_n(p) = (1 - p^2)p^n, \quad n \in \mathbb{N}, \tag{1.16}$$

parametrized by one real parameter $p \in [0, 1)$. The conserved quantities H, Q, and E for the ground state are

$$H(A(p)) = 1, \quad Q(A(p)) = 1, \quad E(A(p)) = \frac{1 + p^2}{1 - p^2}. \tag{1.17}$$

By the ground state *orbit* (for a given p) we shall mean the set obtained from the ground state $A(p)$ by acting on it with the gauge symmetries (1.5) and (1.6):

$$\mathcal{A}(p) = \left\{ \left(e^{i\theta + i\mu n} A_n(p) \right)_{n \in \mathbb{N}} : (\theta, \mu) \in \mathbb{S}^1 \times \mathbb{S}^1 \right\}. \tag{1.18}$$

Our main goal is to show that the ground state is *orbitally stable*; i.e., a perturbed ground state with given p stays close to its orbit $\mathcal{A}(p)$ for all later times provided that the perturbation is small enough. To measure the distance (in some norm X) between the solution and the ground state orbit, we introduce the metric

$$\text{dist}_X(\alpha(t), \mathcal{A}(p)) := \inf_{\theta, \mu \in \mathbb{S}} \| \alpha(t) - e^{i\theta + i\mu \cdot} A(p(t)) \|_X. \tag{1.19}$$

We shall study orbital stability using the Lyapunov method based on the variational formulation (1.12), spectral analysis, and coercivity estimates. The main difficulty is due to the degeneracy of the ground state as a constrained maximizer of energy. To eliminate this degeneracy, we shall use the symplectic orthogonal decomposition, combined with gauge symmetries and conservation laws. In addition to the conservation of $H(\alpha)$, $Q(\alpha)$, and $E(\alpha)$, we shall use the following independent conserved quantity

$$Z(\alpha) = \sum_{n=0}^{\infty} (n + 1)(n + 2)\bar{\alpha}_{n+1}\alpha_n. \tag{1.20}$$

Conservation of $Z(\alpha)$ in the time evolution of (1.1) is proven in Appendix A.

For $p = 0$ the ground state reduces to the single-mode state

$$A_n(0) = \delta_{n0}, \quad n \in \mathbb{N}, \tag{1.21}$$

and the symmetry orbit (1.18) becomes one-dimensional. We will show that in this case a two-parameter orthogonal decomposition involving the generators of scaling (1.4) and gauge (1.5) symmetries suffices to eliminate the degeneracy. Having that and using the conservation laws, we establish the orbital stability of the single-mode state, as stated in the following theorem.

THEOREM 1.3. *For every small $\epsilon > 0$, there is $\delta > 0$ such that for every initial data $\alpha(0) \in h^1$ with $\|\alpha(0) - A(0)\|_{h^1} \leq \delta$, the corresponding unique solution $\alpha(t) \in C(\mathbb{R}, h^1)$ of* (1.1) *satisfies for all t:*

$$\operatorname{dist}_{h^1}(\alpha(t), \mathcal{A}(0)) \leq \epsilon. \tag{1.22}$$

For $p \in (0,1)$, we need to introduce the four-parameter orthogonal decomposition involving both gauge symmetries (1.5) and (1.6), the scaling symmetry (1.4), and the parameter p itself. Having that and using the variational characterization of the constrained maximizers in (1.12), together with the conservation of $E(\alpha)$ and $Z(\alpha)$, we are able to eliminate the degeneracy of the ground state and to prove the following nonlinear orbital stability result:

THEOREM 1.4. *For every $p_0 \in (0,1)$ and every small $\epsilon > 0$, there is $\delta > 0$ such that for every initial data $\alpha(0) \in h^1$ satisfying $\|\alpha(0) - A(p_0)\|_{h^1} \leq \delta$, the corresponding unique solution $\alpha(t) \in C(\mathbb{R}, h^1)$ of* (1.1) *satisfies for all t*:

$$\operatorname{dist}_{h^1}(\alpha(t), \mathcal{A}(p_0)) \leq \epsilon. \tag{1.23}$$

Although it is not needed for the orbital stability, we shall give an explicit description of the spectrum of the linearized operator around the ground state. We believe that the results of this spectral analysis are interesting on their own. In particular, some intriguing algebraic properties of the spectrum may provide a hint in searching for a Lax pair for the conformal flow.

The conformal flow system (1.1) is structurally similar to the Fourier representations of the cubic Szegő equation [5–7] and the lowest Landau level (LLL) equation [1, 8, 9]. These two equations also possess ground states that saturate inequalities analogous to (1.15). Their nonlinear orbital stabilities were established in [5] and [4, 8], respectively, by compactness-type arguments. Such arguments are shorter compared to the Lyapunov approach; however, we prefer the latter because it gives a hands-on control of the perturbations and, more importantly, can also be applied to *local* constrained minimizers (or maximizers).

Let us explain why the proof of nonlinear orbital stability of the ground state is more difficult in our case. The ground state of the cubic Szegő equation is a maximizer of energy under *two* constraints [5] (rather than one, as in our case). On the other hand, the ground state of the LLL equation is the maximizer of energy under one constraint (as in our case), but the ground state can be continued with respect to parameter p by an action of the symmetry transformation (namely, the magnetic translation) [4, 8]. We eliminate the drift of the ground state with respect to parameter p by using the conserved quantities $E(\alpha)$ and $Z(\alpha)$.

The paper is organized as follows. Theorems 1.1 and 1.2 are proved in Sections 2 and 3, respectively. Section 4 introduces the second variation of the energy function and the spectral stability problem. Sections 5 and 6 are devoted to the spectral stability analysis of the single-mode states and the ground state. Theorems 1.3 and 1.4 are proved in Sections 7 and 8, respectively. Appendix A gives the proof of conservation of $Z(\alpha)$. Appendix B summarizes some summation identities that are used.

2 Global Solutions to the Cauchy Problem

The global well-posedness of the conformal flow for h^1 initial data, stated in Theorem 1.1, is a simple corollary of the following local well-posedness result and the conservation of E.

LEMMA 2.1. *For every initial data $\alpha(0) \in h^1$, there exists time $T > 0$ and a unique solution $\alpha(t) \in C((-T,T),h^1)$ of* (1.1). *Moreover, the map $\alpha(0) \mapsto \alpha(t)$ is continuous in h^1 for every $t \in (-T,T)$.*

PROOF. Let us rewrite the system (1.1) in the integral form

$$\alpha(t) = \alpha(0) - i \int_0^t F(\alpha(\tau)) d\tau, \tag{2.1}$$

where the vector field F is given by

$$[F(\alpha)]_n := \sum_{j=0}^{\infty} \sum_{k=0}^{n+j} \frac{\min(n,j,k,n+j-k)+1}{n+1} \bar{\alpha}_j \alpha_k \alpha_{n+j-k}. \tag{2.2}$$

Let us show that

$$\|F(\alpha)\|_{h^1} \lesssim \|\alpha\|_{h^1}^3. \tag{2.3}$$

Indeed, by using the Fourier transform

$$u(\theta) = \sum_{n \in \mathbb{N}} |\alpha_n| e^{in\theta}, \quad \theta \in \mathbb{S},$$

and the inequality

$$\frac{\min(n,j,k,n+j-k)+1}{n+1} \le 1,$$

we obtain

$$\begin{aligned} \|F(\alpha)\|_{h^1} &\le \left\| \sum_{j=0}^{\infty} \sum_{k=0}^{n+j} |\alpha_j||\alpha_k||\alpha_{n+j-k}| \right\|_{h^1} \\ &\lesssim \|u^3\|_{H^1(\mathbb{S})} \lesssim \|u\|_{H^1(\mathbb{S})}^3 \lesssim \|\alpha\|_{h^1}^3, \end{aligned}$$

where we have used the fact that $H^1(\mathbb{S})$ is a Banach algebra with respect to pointwise multiplication. From (2.3) it follows by the Picard method based on the fixed-point argument on a small time interval $(-T,T)$ that there exists a local solution to the integral equation (2.1) for every initial data $\alpha(0) \in h^1$ and the mapping $\alpha(0) \mapsto \alpha(t)$ is continuous. The time T is inversely proportional to $\|\alpha(0)\|_{h^1}^2$. □

Conservation of H, Q, and E follow from the symmetries of the conformal flow system (1.1). Since the squared norm $\|\alpha(t)\|_{h^1}^2 = E(\alpha(t))$ is conserved in time, the lifespan T in Lemma 2.1 can be extended to infinity, which concludes the proof of Theorem 1.1.

Remark 2.2. The global well-posedness result in Theorem 1.1 can be extended to the spaces h^s with $s > \frac{1}{2}$ and quite possibly even to the critical space $h^{1/2}$, in analogy to the global well-posedness result for the cubic Szegő equation (see theorem 2.1 in [5]). However, the result of Theorem 1.1 is sufficient for our purposes because the orbital stability analysis relies on the global solution in h^1 only.

3 Global Energy Bound

The following lemma establishes the inequality (1.15) and shows that it is saturated only for the geometric sequences. Theorem 1.2 is the reinstatement of the same result.

LEMMA 3.1. *For every complex-valued sequence $A \in h^{1/2}$ the following inequality holds*:

$$\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{n+j}[\min(n,j,k,n+j-k)+1]\bar{A}_n\bar{A}_j A_k A_{n+j-k} \leq \left[\sum_{n=0}^{\infty}(n+1)|A_n|^2\right]^2. \tag{3.1}$$

Moreover, this inequality is saturated if and only if $A_n = cp^n$ for $c, p \in \mathbb{C}$ with $|p| < 1$.

PROOF. The left-hand side of (3.1) can be written as

$$\sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{k=0}^{n}[\min(j,n-j,k,n-k)+1]\bar{A}_j\bar{A}_{n-j}A_k A_{n-k}, \tag{3.2}$$

whereas the right-hand side of (3.1) can be written as

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}(k+1)(n+1-k)|A_k|^2|A_{n-k}|^2. \tag{3.3}$$

Let us fix $n \in \mathbb{N}$ and denote $x_k := A_k A_{n-k}$ for $0 \leq k \leq n$. To prove the inequality (3.1), it suffices to show that the following quadratic form is nonnegative:

$$\sum_{k=0}^{n}(k+1)(n+1-k)|x_k|^2 - \sum_{j=0}^{n}\sum_{k=0}^{n}[\min(j,n-j,k,n-k)+1]\bar{x}_j x_k.$$

Let us prove this for odd $n = 2N+1$ with $N \in \mathbb{N}$. The proof for even n is analogous. Since $x_k = x_{n-k}$ by definition, we can simplify the sums (3.2) and

(3.3) as follows:

$$\sum_{j=0}^{n}\sum_{k=0}^{n}[\min(j,n-j,k,n-k)+1]\overline{x}_j x_k = 4\sum_{j=0}^{N}\sum_{k=0}^{N}[\min(j,k)+1]\overline{x}_j x_k. \tag{3.4}$$

and

$$\sum_{k=0}^{n}(k+1)(n+1-k)|x_k|^2 = 2\sum_{k=0}^{N}(k+1)(2N+2-k)|x_k|^2. \tag{3.5}$$

Subtracting (3.4) from (3.5) we obtain the following identity:

$$\begin{aligned} &2\sum_{k=0}^{N}(k+1)(2N+2-k)|x_k|^2 - 4\sum_{j=0}^{N}\sum_{k=0}^{N}[\min(j,k)+1]\overline{x}_j x_k \\ &\quad = 4\sum_{j=0}^{N-1}\sum_{k=j+1}^{N}(j+1)|x_j - x_k|^2 \geq 0. \end{aligned} \tag{3.6}$$

The identity (3.6) is proven by induction in N. The case $N = 0$ is trivial. The case $N = 1$ is verified by inspection. For general N, the difference between the left-hand sides of (3.6) evaluated at $N+1$ and N is

$$\begin{aligned} &4\sum_{k=0}^{N}(k+1)\big(|x_k|^2 - \overline{x}_{N+1}x_k - x_{N+1}\overline{x}_k\big) + 2(N+1)(N+2)|x_{N+1}|^2 \\ &\quad = 4\sum_{k=0}^{N}(k+1)|x_k - x_{N+1}|^2, \end{aligned}$$

which is equal to the difference between the right-hand sides of (3.6) evaluated at $N+1$ and N. By induction, the identity (3.6), which holds for $N = 0, 1$, will hold for any $N \in \mathbb{N}$.

Combining (3.6) with a similar result for $n = 2N$ and summing up with respect to $N \in \mathbb{N}$, we obtain the inequality (3.1). The inequality is saturated when the double sum in (3.6) vanishes, which happens if $x_k = A_k A_{n-k}$ is independent of k for every $0 \leq k \leq n$ (but may depend on n for $n \in \mathbb{N}$). This is true if and only if $A_k = cp^k$ for some $c, p \in \mathbb{C}$. The constraint $|p| < 1$ is needed since $A \in h^{1/2}$. □

Remark 3.2. Theorem 1.2 provides an alternative variational characterization of the ground state family (1.14). Indeed, from Theorem 1.2, we know that $G := Q^2 - H$ attains the global minimum equal to zero at the geometric sequence $A_n = cp^n$; hence its first variation at A vanishes

$$G'(A) = 2Q(A)Q'(A) - H'(A) = 0. \tag{3.7}$$

Comparing this with the variational characterization (1.12) we see that $A_n = cp^n$ is a solution of (1.11) with $\lambda = Q(A) = |c|^2/(1-|p|^2)^2$.

4 Second Variation and the Spectral Stability

Here we compute the second variation of the functional K, defined in (1.12), at a stationary solution and use this result to formulate the spectral stability problem.

Let $\alpha = A + a + ib$, where A is a real root of the algebraic system (1.11), whereas a and b are real and imaginary parts of the perturbation. Because the stationary solution A is a critical point of K, the first variation of K vanishes at $\alpha = A$, and the second variation of K at $\alpha = A$ can be written as a quadratic form associated with the Hessian operator. In variables above, we obtain the quadratic form in the diagonalized form:

$$K(A+a+ib) - K(A) = \langle L_+ a, a\rangle + \langle L_- b, b\rangle + \mathcal{O}(\|a\|^3 + \|b\|^3), \tag{4.1}$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $\ell^2(\mathbb{N})$ and $\|\cdot\|$ is the induced norm.

After straightforward computations, we obtain the explicit form for the self-adjoint operators $L_\pm$: $D(L_\pm) \to \ell^2$, where $D(L_\pm) \subset \ell^2$ is the maximal domain of the unbounded operators $L_\pm$. After the interchange of summations, we obtain

$$(L_\pm a)_n = (B_\pm a)_n - (n+1)\lambda a_n, \tag{4.2}$$

where

$$(B_\pm a)_n = \sum_{j=0}^{\infty} \left[2 \sum_{k=\max(0,j-n)}^{\infty} S_{njk,n+k-j} A_k A_{n+k-j} \right.$$
$$\left. \pm \sum_{k=0}^{n+j} S_{njk,n+j-k} A_k A_{n+j-k} \right] a_j .$$

The Hessian operator given by the self-adjoint operators $L_\pm$ also defines the linearized stability problem for the small perturbations to the stationary states. Let us consider the following decomposition of solutions of the conformal flow (1.1),

$$\alpha(t) = e^{-i\lambda t}[A + a(t) + ib(t)], \tag{4.3}$$

with real A, a, and b. When the nonlinear system (1.1) is truncated at the linearized approximation with respect to a and b, we obtain the linearized evolution system

$$M\frac{da}{dt} = L_- b, \quad M\frac{db}{dt} = -L_+ a, \tag{4.4}$$

where $M = \mathrm{diag}(1, 2, \dots)$ is the diagonal matrix operator of positive integers. Substituting $a(t) = e^{\Lambda t}\mathtt{a}$ and $b(t) = e^{\Lambda t}\mathtt{b}$ into (4.4), we get the eigenvalue problem

$$L_- \mathtt{a} = \Lambda M \mathtt{b}, \quad L_+ \mathtt{b} = -\Lambda M \mathtt{a}. \tag{4.5}$$

We say that the stationary solution A is *spectrally stable* if all the eigenvalues Λ lie on the imaginary axis.

5 Spectral Stability of the Single-Mode States

Here we compute $L_\pm$ and eigenvalues of the spectral stability problem (4.5) for the single-mode states (1.13). Because of the scaling transformation (1.4), we set $c = 1$ and hence $\lambda = 1$. In this case, the self-adjoint operators $L_\pm : D(L_\pm) \to \ell^2$ are given explicitly by

$$(L_\pm a)_n = [2\min(n, N) + 1 - n]a_n \pm [\min(n, N, 2N - n) + 1]a_{2N-n}, \quad n \in \mathbb{N}, \tag{5.1}$$

from which it is clear that $D(L_\pm) = h^1$.

We prove the following characterization of eigenvalues of $L_\pm$, from which we obtain the spectral stability of the single-mode states (1.13).

LEMMA 5.1. *The single-mode state* (1.13) *with* $N \in \mathbb{N}$ *is a degenerate saddle point of* K *with* $2N + 1$ *positive eigenvalues* (*counted with multiplicities*)*, the zero eigenvalue of multiplicity* $2N + 3$*, and infinitely many negative eigenvalues bounded away from zero.*

PROOF. The operators $L_\pm$ in (5.1) consist of a $(2N + 1) \times (2N + 1)$ block denoted by $\widetilde{L}_\pm$ and a diagonal block with entries $\{2N + 1 - n\}_{n\geq 2N+1}$. The latter diagonal block has one zero eigenvalue and all other eigenvalues are strictly negative. The former block can be written in the form

$$\widetilde{L}_+ = \begin{bmatrix} L_{11} & 0 & L_{12} \\ 0 & 2(N+1) & 0 \\ L_{12}^T & 0 & L_{22} \end{bmatrix}, \quad \widetilde{L}_- = \begin{bmatrix} L_{11} & 0 & -L_{12} \\ 0 & 0 & 0 \\ -L_{12}^T & 0 & L_{22} \end{bmatrix},$$

where

$$\begin{aligned} L_{11} &= \mathrm{diag}(1, 2, \ldots, N), \\ L_{12} &= \mathrm{antidiag}(1, 2, \ldots, N), \\ L_{22} &= \mathrm{diag}(N, N - 1, \ldots, 1). \end{aligned}$$

Since the eigenvalue problem for $\widetilde{L}_\pm$ decouples into N pairs and one equation, we can compute the eigenvalues of $\widetilde{L}_\pm$. The block $\widetilde{L}_+$ has $N + 1$ positive eigenvalues and the zero eigenvalue of multiplicity N. The block $\widetilde{L}_-$ has N positive eigenvalues and the zero eigenvalue of multiplicity $N + 1$. The assertion of the lemma follows by combining the count of all eigenvalues. □

LEMMA 5.2. *The single-mode state* (1.13) *with* $N \in \mathbb{N}_+$ *module to the gauge symmetry* (1.5) *is a degenerate saddle point of* H *under fixed* Q *with* $2N$ *positive eigenvalues* (*counted with multiplicities*)*, the zero eigenvalue of multiplicity* $2N+2$*, and infinitely many negative eigenvalues bounded away from zero. The single-mode state* (1.13) *with* $N = 0$ *module to the gauge symmetry* (1.5) *is a degenerate maximizer of* H *under fixed* Q *with a double zero eigenvalue.*

PROOF. The count of eigenvalues in Lemma 5.1 is modified by two constraints as follows. One positive eigenvalue corresponds to the central entry $2(N+1)$ in $\widetilde{L}_+$. This positive eigenvalue is removed by the constraint of fixed $Q(\alpha)$ in the variational formulation (1.12). Indeed, if $\alpha = A + a + ib$ and $Q(\alpha)$ is fixed, we impose the linear constraint on the real perturbation a in the form

$$\langle MA, a\rangle = \sum_{j=0}^{\infty}(j+1)A_j a_j = 0, \tag{5.2}$$

which yields the constraint $a_N = 0$ for the single-mode state (1.13). This constraint removes the corresponding positive entry of $\widetilde{L}_+$.

Similarly, the zero eigenvalue from the central zero entry in $\widetilde{L}_-$ corresponds to the gauge symmetry (1.5). In order to define uniquely the parameter θ due to the gauge symmetry (1.5), we impose the linear constraint on the perturbation b in the form

$$\langle A, b\rangle = \sum_{j=0}^{\infty} A_j b_j = 0, \tag{5.3}$$

which yields the constraint $b_N = 0$ for the single-mode state (1.13). This constraint removes the corresponding zero entry of $\widetilde{L}_-$. □

LEMMA 5.3. *All single-mode states* (1.13) *are spectrally stable.*

PROOF. Due to the block diagonalization of $L_\pm$, one can solve the spectral stability problem (4.5) explicitly. Associated to the diagonal blocks of $L_\pm$, we obtain an infinite sequence of eigenvalues $\Lambda_n = \pm i\Omega_n$, where

$$\Omega_n = \frac{n-2N-1}{n+1}, \quad n \geq 2N+1. \tag{5.4}$$

Note that $\Omega_{2N+1} = 0$ corresponds to the zero eigenvalue of $L_\pm$ associated with this entry, whereas $\Omega_n > 0$ for $n \geq 2N+2$ corresponds to the negative eigenvalues of $L_\pm$ in the corresponding entries.

For the $(2N+1)\times(2N+1)$ block of $L_\pm$ denoted by $\widetilde{L}_\pm$, we obtain a sequence of N eigenvalues $\Lambda_n = \pm i\Omega_n$, where

$$\Omega_n = \frac{2(N-n)}{2N+1-n}, \quad 0 \leq n \leq N-1. \tag{5.5}$$

These eigenvalues correspond to the positive eigenvalues of $L_\pm$. In addition, we count the zero eigenvalue $\Omega = 0$ of geometric multiplicity $2N+1$ and algebraic multiplicity $2N+2$ in the spectral problem (4.5) associated with the same block $\widetilde{L}_\pm$. No other eigenvalues exist; hence for every $N \in \mathbb{N}$ the single-mode state (1.13) is spectrally stable. □

Remark 5.4. All eigenvalues of the spectral problem (4.5) for the single-mode state (1.13) with any $N \in \mathbb{N}$ are semisimple except for the double zero eigenvalue related to the gauge symmetry (1.5).

6 Spectral Stability of the Ground State

In the case of the ground state (1.16), the self-adjoint operators $L_\pm : D(L_\pm) \to \ell^2$ defined by (4.2) take the following form

$$[L_\pm(p)a]_n = \sum_{j=0}^{\infty} [B_\pm(p)]_{nj} a_j - (n+1)a_n, \tag{6.1}$$

where

$$\begin{aligned} [B_\pm(p)]_{nj} &= (1-p^2)^2 \Bigg[2 \sum_{k=\max(0,j-n)}^{\infty} S_{njk,n+k-j}\, p^{n+2k-j} \\ &\qquad \pm \sum_{k=0}^{n+j} S_{njk,n+j-k}\, p^{n+j} \Bigg] \\ &= 2p^{|n-j|} - 2p^{2+n+j} \pm (1-p^2)^2 (j+1)(n+1) p^{n+j}. \end{aligned} \tag{6.2}$$

To derive this expression we have used relations (B.4)–(B.6) from Appendix B. Note that the first term in $B_\pm(p)$ is given by the Toeplitz operator, while the second and third terms in $B_\pm(p)$ are given by outer products.

First, we establish commutativity of the linear operators given by (6.1)–(6.2).

LEMMA 6.1. *For every $p \in [0,1)$, we have*

$$[L_+(p), L_-(p)] = 0 \tag{6.3}$$

and

$$[M^{-1}L_+(p), M^{-1}L_-(p)] = 0. \tag{6.4}$$

PROOF. In order to verify the commutation relation (6.3), we use (6.1)–(6.2) and write

$$[L_+(p)L_-(p) - L_-(p)L_+(p)]_{nj} = 2(1-p^2)^2 C_{nj}, \tag{6.5}$$

where

$$\begin{aligned} C_{nj} = \sum_{k=0}^{\infty} & (1+k)(1+n)p^{n+k}\left[2p^{|k-j|} - 2p^{k+j+2} - (k+1)\delta_{kj}\right] \\ & - (1+k)(1+j)p^{k+j}\left[2p^{|k-n|} - 2p^{k+n+2} - (k+1)\delta_{kn}\right]. \end{aligned}$$

To show that $C_{nj} = 0$, we proceed for $j \geq n$ (the proof for $j \leq n$ is analogous):

$$\begin{aligned} C_{nj} = {} & 2\sum_{k=0}^{\infty}(1+k)(1+n)p^{n+k+|k-j|} - 2\sum_{k=0}^{\infty}(1+k)(1+j)p^{j+k+|k-n|} \\ & + 2\sum_{k=0}^{\infty}(1+k)(j-n)p^{n+j+2k+2} + (1+n)(1+j)(n-j)p^{n+j} \end{aligned}$$

such that

$$\begin{aligned} C_{nj} &= 2\sum_{k=0}^{j}(1+k)(1+n)p^{n+j} + 2\sum_{k=j+1}^{\infty}(1+k)(1+n)p^{n+2k-j} \\ &\quad - 2\sum_{k=0}^{\infty}(1+k)(1+j)p^{n+j} - 2\sum_{k=n+1}^{\infty}(1+k)(1+j)p^{j+2k-n} \\ &\quad + 2\sum_{k=0}^{\infty}(1+k)(j-n)p^{n+j+2k+2} + (1+n)(1+j)(n-j)p^{n+j}. \end{aligned}$$

This yields $C_{nj} = 0$ in view of identities (B.1)–(B.3) from Appendix B.

In order to verify the commutation relation (6.4), we use (6.1)–(6.2) and write

$$\left[L_+(p)M^{-1}L_-(p) - L_-(p)M^{-1}L_+(p)\right]_{nj} = 2(1-p^2)^2 D_{nj}, \tag{6.6}$$

where

$$\begin{aligned} D_{nj} &= \sum_{k=0}^{\infty}(1+n)p^{n+k}\left[2p^{|k-j|} - 2p^{k+j+2} - (k+1)\delta_{kj}\right] \\ &\qquad - (1+j)p^{k+j}\left[2p^{|k-n|} - 2p^{k+n+2} - (k+1)\delta_{kn}\right]. \end{aligned}$$

As previously, we proceed for $j \geq n$ (the proof for $j \leq n$ is analogous):

$$\begin{aligned} D_{nj} &= 2\sum_{k=0}^{\infty}(1+n)p^{n+k+|k-j|} - 2\sum_{k=0}^{\infty}(1+j)p^{j+k+|k-n|} \\ &\quad + 2\sum_{k=0}^{\infty}(j-n)p^{n+j+2k+2} \\ &= 2\sum_{k=j+1}^{\infty}(1+n)p^{n+2k-j} - 2\sum_{k=n+1}^{\infty}(1+j)p^{j+2k-n} \\ &\quad + 2\sum_{k=0}^{\infty}(j-n)p^{n+j+2k+2}. \end{aligned}$$

This yields $D_{nj} = 0$ in view of the identity (B.1) from Appendix B. □

Because of the commutation relation (6.3), the operators $L_\pm(p)$ have a common basis of eigenvectors. The following lemma fully characterizes the spectra of $L_\pm(p)$ in ℓ^2.

LEMMA 6.2. *For every* $p \in [0,1)$*, the spectra of the operators* $L_\pm(p) : h^1 \subset \ell^2 \to \ell^2$ *given by* (6.1) *consist of the following isolated eigenvalues*:

$$\sigma(L_-) = \{\ldots, -3, -2, -1, 0, 0\}, \tag{6.7}$$

and

$$\sigma(L_+) = \{\dots, -3, -2, -1, 0, \lambda_*(p)\}, \tag{6.8}$$

where $\lambda_*(p) = 2(1+p^2)/(1-p^2) > 0$.

PROOF. For every $p \in [0,1)$ we have $A(p) \in h^m$ for every $m \in \mathbb{N}$. Hence $B_\pm(p)$ are bounded operators from ℓ^2 to ℓ^2, whereas the diagonal parts of $L_\pm(p)$ are unbounded operators from ℓ^2 to ℓ^2 with the domain h^1. Since the diagonal parts of $L_\pm(p)$ (of the Hilbert-Schmidt type) have compact resolvent and the $B_\pm(p)$ operators are bounded perturbations to the diagonal parts, the operators $L_\pm(p)$ have compact resolvent. Hence, the spectra of $L_\pm(p)$ consist of infinitely many isolated eigenvalues.

By the symmetries (1.5) and (1.6), we have

$$L_-(p)A(p) = 0, \quad L_-(p)MA(p) = 0, \tag{6.9}$$

where $M = \mathrm{diag}(1, 2, \dots)$. On the other hand, by differentiating (1.16) with respect to p, we obtain

$$L_+A'(p) = 0 \quad \text{where } A'(p) = -\frac{(1+p^2)}{p(1-p^2)}A(p) + \frac{1}{p}MA(p). \tag{6.10}$$

By differentiating the scaling symmetry (1.4) with respect to c at $c = 1$, we get the explicit solution $a = A(p)$, $b = -2tA(p)$ of the linear equation (4.4), hence

$$L_+(p)A(p) = 2MA(p). \tag{6.11}$$

Combined with (6.10), this yields

$$L_+(p)MA(p) = \frac{1+p^2}{1-p^2}L_+(p)A(p) = \frac{2(1+p^2)}{1-p^2}MA(p), \tag{6.12}$$

which gives the positive eigenvalue $\lambda_*(p)$ in (6.8).

It remains to prove that the rest of the spectrum of $L_\pm(p)$ coincides with the set of negative integers. Since the same two-dimensional subspace

$$X_0(p) = \mathrm{span}\{A(p), MA(p)\}$$

is associated with the double zero eigenvalue of operator $L_-(p)$ and with the two simple nonnegative eigenvalues of operator $L_+(p)$, we introduce the orthogonal complement

$$[X_0(p)]^\perp := \{a \in \ell^2 \colon \langle A(p), a\rangle = \langle MA(p), a\rangle = 0\}. \tag{6.13}$$

Eigenvectors for negative eigenvalues of $L_\pm(p)$ belong to $[X_0(p)]^\perp$. Due to the second orthogonality condition in (6.13), we compute

$$\begin{aligned}[B_\pm(p)a]_n = \sum_{j\in\mathbb{N}} \big[&2p^{|n-j|} - 2p^{2+n+j} \\ &\pm (1-p^2)^2(j+1)(n+1)p^{n+j}\big]a_j = 2[T(p)a]_n,\end{aligned} \tag{6.14}$$

where $[T(p)]_{nj} := p^{|n-j|} - p^{n+j+2}$. Hence, the negative eigenvalues of $L_\pm(p)$ are identical to the negative eigenvalues of $2T(p)-M$, and hence they are identical to each other.

In order to prove that the negative eigenvalues of $2T(p) - M$ are negative integers, let us define the shift operator $S\colon \ell^2 \to \ell^2$ by

$$S\colon (a_0, a_1, a_2, \dots) \mapsto (0, a_0, a_1, a_2, \dots),$$

and its left inverse operator $S^*\colon \ell^2 \to \ell^2$ by

$$S^*\colon (a_0, a_1, a_2, \dots) \mapsto (a_1, a_2, a_3, \dots).$$

Let us show that for every $a \in [X_0(p)]^\perp$,

$$[2T(p) - M, S]a = -Sa, \quad [2T(p) - M, S^*]a = S^*a. \tag{6.15}$$

Indeed, the first identity in (6.15) is verified if the following two expressions are equal to each other:

$$\begin{aligned}[(2T(p) - M)Sa]_n &= 2\sum_{j=1}^{\infty}(p^{|n-j|} - p^{2+n+j})a_{j-1} - (n+1)a_{n-1} \\ &= 2\sum_{k=0}^{\infty}(p^{|n-1-k|} - p^{3+n+k})a_k - (n+1)a_{n-1}\end{aligned}$$

and

$$\begin{aligned}[S(2T(p) - M - I)a]_n &= [(2T(p) - M - I)a]_{n-1} \\ &= 2\sum_{k=0}^{\infty}(p^{|n-1-k|} - p^{1+n+k})a_k - (n+1)a_{n-1}.\end{aligned}$$

Since $\sum_{k=0}^{\infty} p^k a_k = 0$ thanks to the first orthogonality condition in (6.13), the two expressions are equal to each other so that the first identity in (6.15) is verified. Now, the second identity in (6.15) is verified if the following two expressions are equal to each other:

$$\begin{aligned}[(2T(p) - M)S^*a]_n &= 2\sum_{j=0}^{\infty}(p^{|n-j|} - p^{2+n+j})a_{j+1} - (n+1)a_{n+1} \\ &= 2\sum_{k=1}^{\infty}(p^{|n+1-k|} - p^{1+n+k})a_k - (n+1)a_{n+1} \\ &= 2\sum_{k=0}^{\infty}(p^{|n+1-k|} - p^{1+n+k})a_k - (n+1)a_{n+1}\end{aligned}$$

and

$$[S^*(2T(p) - M + I)a]_n = [(2T(p) - M + I)a]_{n+1}$$
$$= 2\sum_{k=0}^{\infty}(p^{|n+1-k|} - p^{3+n+k})a_k - (n+1)a_{n+1}.$$

The two expressions are equal to each other again thanks to the first orthogonality condition in (6.13), so that the second identity in (6.15) is verified.

The two identities in (6.15) imply that the operators S and S^* play the role of creation and annihilation operators for elements of $[X_0(p)]^{\perp}$. In particular, they generate the (same) set of eigenvectors of $L_{\pm}(p)$ for the (same) eigenvalues of $L_{\pm}(p)$. More precisely, the second equation in (6.15) shows that the negative eigenvalues of $2T(p) - M$ are located at negative integers, $\lambda_m = -m$, $m \in \mathbb{N}_+$, whereas the corresponding eigenvectors $v^{(m)}$, defined by

$$L_{\pm}(p)v^{(m)} = (2T(p) - M)v^{(m)} = \lambda_m v^{(m)}, \quad m \in \mathbb{N}_+,$$

are related by

$$S^* v^{(m)}(p) = v^{(m-1)}(p), \quad m \geq 2,$$

and

$$S^* v^{(1)}(p) \in \ker(2T(p) - M) = \ker(L_+(p)) = \operatorname{span}\{A'(p)\}.$$

The first equation in (6.15) gives the relation

$$v^{(m+1)}(p) = S v^{(m)}(p), \quad m \geq 1,$$

which can be used to generate all eigenvectors from $v^{(1)}(p)$. Using summation formulae (B.7)–(B.8) from Appendix B, we have verified that the first eigenvector is given by

$$[v^{(1)}(p)]_n = \begin{cases} p^2 & \text{if } n = 0, \\ (1-p^2)[-(1+p^2) + n(1-p^2)]p^{n-2} & \text{if } n \in \mathbb{N}_+, \end{cases} \tag{6.16}$$

so that $S^* v^{(1)}(p) = (1-p^2)A'(p) \in \ker(L_+(p)) = \ker(2T(p) - M)$. □

Because of the commutation relation (6.4), the operators $M^{-1}L_{\pm}(p)$ also have a common basis of eigenvectors, which coincide with eigenvectors of the spectral problem (4.5) for nonzero eigenvalues Λ. The following lemma fully characterizes eigenvalues of the spectral problem (4.5) in ℓ^2.

LEMMA 6.3. *Eigenvalues of the spectral problem* (4.5) *are purely imaginary* $\Lambda_m = \pm i\Omega_m$, *where*

$$\Omega_0 = \Omega_1 = 0, \quad \Omega_m = \frac{m-1}{m+1}, \quad m \geq 2, \tag{6.17}$$

independently of $p \in (0,1)$.

PROOF. The spectral problem (4.5) can be written in the matrix form

$$\mathcal{L}(p)\vec{\mathrm{a}} = \Lambda \mathcal{M}\vec{\mathrm{a}}, \tag{6.18}$$

where

$$\mathcal{L}(p) = \begin{bmatrix} 0 & L_-(p) \\ -L_+(p) & 0 \end{bmatrix}, \quad \mathcal{M} = \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}, \quad \vec{\mathrm{a}} = \begin{bmatrix} \mathrm{a} \\ \mathrm{b} \end{bmatrix}.$$

The geometric kernel of $\mathcal{L}(p)$ is three-dimensional and spanned by the three linearly independent eigenvectors

$$\begin{bmatrix} 0 \\ A(p) \end{bmatrix}, \quad \begin{bmatrix} 0 \\ MA(p) \end{bmatrix}, \quad \begin{bmatrix} A'(p) \\ 0 \end{bmatrix}, \tag{6.19}$$

according to (6.9) and (6.10). The generalized kernel of $\mathcal{M}^{-1}\mathcal{L}(p)$ is obtained from solutions of the inhomogeneous equation

$$\mathcal{L}(p)\vec{\mathrm{a}}_1 = \mathcal{M}\vec{\mathrm{a}}_0, \tag{6.20}$$

where $\vec{\mathrm{a}}_0 \in \ker(\mathcal{L}(p))$. We have

$$\langle MA'(p), A(p)\rangle = 0, \quad \langle MA'(p), MA(p)\rangle \neq 0, \tag{6.21}$$

thanks to equations (6.11), (6.12), and the explicit computation

$$\begin{aligned} \langle MA'(p), MA(p)\rangle &= \sum_{n=0}^{\infty} (n+1)^2 p^{2n}[n(1-p^2) - 2p^2] \\ &= \frac{2p^2}{(1-p^2)^3} > 0. \end{aligned} \tag{6.22}$$

Therefore, the Jordan blocks are simple for the second and third eigenvectors in (6.19) and at least double for the first eigenvector in (6.19). Indeed, the following generalized eigenvector $\vec{\mathrm{a}}_1$ follows from (6.11) and satisfies (6.20) with the eigenvector $\vec{\mathrm{a}}_0$, where

$$\vec{\mathrm{a}}_1 = -\frac{1}{2}\begin{bmatrix} A(p) \\ 0 \end{bmatrix}, \quad \vec{\mathrm{a}}_0 = \begin{bmatrix} 0 \\ A(p) \end{bmatrix}. \tag{6.23}$$

The corresponding Jordan block is exactly double because $MA(p)$ is not orthogonal to $\ker(L_-(p))$. Hence the zero eigenvalue has multiplicity 4 with three eigenvectors in (6.19) and one generalized eigenvector in (6.23).

It remains to study the nonzero eigenvalues of the spectral problem (4.5). To do so, we study negative eigenvalues of the commuting operators $M^{-1}L_\pm(p)$. Due to the presence of the operators M, we introduce a different complement of the two-dimensional subspace $X_0(p) = \operatorname{span}\{A(p), MA(p)\}$ compared to (6.13). Namely, we define

$$[X_c(p)]^\perp := \{a \in \ell^2\colon \langle MA(p), a\rangle = \langle M^2 A(p), a\rangle = 0\}. \tag{6.24}$$

Eigenvectors for negative eigenvalues of $M^{-1}L_\pm(p)$ belong to $[X_c(p)]^\perp$. Due to the first orthogonality condition in (6.24), we have $B_\pm(p)a = 2T(p)a$ as in

(6.14). Hence, the negative eigenvalues of $M^{-1}L_\pm(p)$ are identical to the negative eigenvalues of $2M^{-1}T(p) - I$, and hence they are identical to each other. The spectral problem $L_\pm v = \lambda M v$ for $\lambda < 0$ can be rewritten in the equivalent form

$$T(p)v = \mu M v, \quad \mu := \frac{1+\lambda}{2}. \tag{6.25}$$

Let us prove that the spectral problem (6.25) admit a countable set of eigenvalues

$$\mu_m = \frac{1}{m+1}, \quad m \in \mathbb{N}, \tag{6.26}$$

with the corresponding eigenvectors given by

$$v^{(m)} = M^m A(p) - \sum_{j=0}^{m-1} \alpha_j^{(m)}(p) M^j A(p), \tag{6.27}$$

where the coefficients $\{\alpha_j^{(m)}(p)\}_{j=0}^{m-1}$ are uniquely found from orthogonality conditions

$$\langle M^j A(p), v^{(m)} \rangle = 0, \quad 1 \le j \le m.$$

Consequently, $v^{(m)} \in [X_c(p)]^\perp$ for every $m \ge 2$. Indeed, we have the first few eigenvalues and eigenvectors explicitly:

$$\mu_0 = 1: \; v^{(0)} = A(p),$$
$$\mu_1 = \frac{1}{2}: \; v^{(1)} = MA(p) - \frac{1+p^2}{1-p^2} A(p),$$
$$\mu_2 = \frac{1}{3}: \; v^{(2)} = M^2 A(p) + 3\frac{1+p^2}{1-p^2} MA(p) - 2\frac{1+p^2+p^4}{(1-p^2)^2} A(p),$$

where we recognize the same eigenvectors $A(p)$ and $A'(p)$ for the first two (positive and zero) eigenvalues of $M^{-1}L_\pm(p)$. In order to prove (6.26) and (6.27) for every $m \in \mathbb{N}$, we represent

$$\begin{aligned} &[T(p)M^m A(p)]_n \\ &= (1-p^2)\Big[\sum_{j\in\mathbb{N}} (1+j)^m p^{|n-j|} p^j - \sum_{j\in\mathbb{N}} (1+j)^m p^{2+n+2j}\Big] \\ &= \left[\sum_{j=0}^{n} (1+j)^m + \sum_{j=n+1}^{\infty} (1+j)^m p^{2(j-n)} - \sum_{j=0}^{\infty} (1+j)^m p^{2+2j}\right] A_n(p) \\ &= \left[\sum_{k=1}^{n+1} k^m + p^2 \sum_{k=0}^{\infty} [(1+k+1+n)^m - (1+k)^m] p^{2k}\right] A_n(p). \end{aligned} \tag{6.28}$$

The first sum in (6.28) is computed explicitly as

$$\sum_{k=1}^{n+1} k^m = \frac{(n+1)^{m+1}}{m+1} + \frac{1}{2}(n+1)^m + \sum_{j=1}^{[m/2]} \frac{B_{2j}}{2j} \binom{m}{2j-1} (n+1)^{m+1-2j},$$

where B_{2j} are Bernoulli numbers and the last term is either $(n+1)$ or $(n+1)^2$. The last sum in (6.28) is expanded in the binomial formula with the highest term $(n+1)^m$ and the lowest term $(n+1)$. As a result, the right-hand side of (6.28) is written in positive powers of $(n+1)$, namely,

$$T(p)M^m A(p) = \sum_{j=1}^{m+1} \beta_j^{(m)}(p) M^j A(p), \tag{6.29}$$

where $\{\beta_j^{(m)}(p)\}_{j=1}^{m+1}$ are uniquely defined and $\beta_{m+1}^{(m)}(p) = \frac{1}{m+1}$. The balance of the highest term $(n+1)^{m+1}$ yields the eigenvalue $\mu_m = \frac{1}{m+1}$ of the spectral problem (6.25). Now, adding a linear combination of m terms $\{M^j A(p)\}_{j=0}^{m-1}$ to $M^m A(p)$ as in (6.27) and using expressions (6.29) recursively from $j = m-1$ to $j = 0$, we obtain a linear system of m equations for m coefficients $\{\alpha_j^{(m)}(p)\}_{j=0}^{m-1}$. The linear system is associated with a triangular matrix with nonzero diagonal coefficients; hence, it admits a unique solution for $\{\alpha_j^{(m)}(p)\}_{j=0}^{m-1}$. Hence, the validity of (6.26) and (6.27) is proven.

Thanks to the relation between μ and λ in (6.25), we have shown that the negative eigenvalues of $M^{-1}L_{\pm}(p)$ are given by

$$\lambda_m = 2\mu_m - 1 = -\frac{m-1}{m+1}, \quad m \geq 2. \tag{6.30}$$

The common set of eigenvectors of $M^{-1}L_+(p)$ and $M^{-1}L_-(p)$ for negative eigenvalues λ coincides with the set of eigenvectors of the spectral problem (4.5) for nonzero eigenvalues Λ. Thus, the nonzero eigenvalues of the spectral problem (4.5) are given by $\Lambda_m^2 = -\lambda_m^2$, which yields the explicit expression (6.17) thanks to (6.30). □

Remark 6.4. All eigenvalues of the spectral problem (4.5) for the ground state (1.16) are simple except for the zero eigenvalue, which has geometric multiplicity 4 and algebraic multiplicity 3. The p-independent eigenvalues (6.17) coincide with the eigenvalues (5.4) with $N = 0$, since the $N = 0$ single-mode state (1.13) is the limit of the ground state (1.16) as $p \to 0$.

7 Orbital Stability of the $N = 0$ Single-Mode State

Here we prove Theorem 1.3, which states the orbital stability of the $N = 0$ single-mode state (1.13) with the normalization $c = 1$ or $\lambda = 1$. Since this is the limit $p \to 0$ of the ground state (1.16), we will use the expression $A_n(0) = \delta_{n0}$ as in (1.21). In order to prove Theorem 1.3, we decompose a solution of the conformal flow (1.1) into a sum of the two-parameter orbit of the ground state generated by the symmetries (1.4) and (1.5), as well as the symplectically orthogonal remainder term. The following lemma provides a basis for such a decomposition.

LEMMA 7.1. *There exists $\delta_0 > 0$ such that for every $\alpha \in \ell^2$ satisfying*

$$\delta := \inf_{\theta \in \mathbb{S}} \|\alpha - e^{i\theta} A(0)\|_{\ell^2} \leq \delta_0, \tag{7.1}$$

there exists a unique choice of real-valued numbers (c, θ) and real-valued sequences $a, b \in \ell^2$ in the orthogonal decomposition

$$\alpha_n = e^{i\theta}(cA_n(0) + a_n + ib_n), \tag{7.2}$$

subject to the orthogonality conditions

$$\langle MA(0), a\rangle = \langle MA(0), b\rangle = 0, \tag{7.3}$$

satisfying the estimate

$$|c - 1| + \|a + ib\|_{\ell^2} \lesssim \delta. \tag{7.4}$$

PROOF. The proof is based on the inverse function theorem applied to the vector function $F(c, \theta; \alpha) : \mathbb{R}^2 \times \ell^2 \mapsto \mathbb{R}^2$ given by

$$F(c, \theta; \alpha) := \begin{bmatrix} \langle MA(0), \mathrm{Re}(e^{-i\theta}\alpha - cA(0))\rangle \\ \langle MA(0), \mathrm{Im}(e^{-i\theta}\alpha - cA(0))\rangle \end{bmatrix}.$$

The Jacobian matrix DF at $A(0)$ is diagonal and invertible

$$DF(1, 0; A(0)) = -\begin{bmatrix} \langle MA(0), A(0)\rangle & 0 \\ 0 & \langle MA(0), A(0)\rangle \end{bmatrix}.$$

For sufficiently small $\delta > 0$, there exists a unique root (c, θ) near $(1, \theta_0)$ where θ_0 is an argument in the infimum (7.1), with the bound

$$|c - 1| + |\theta - \theta_0| \lesssim \delta.$$

This proves the bound for c in (7.4). By using the definition of (a, b) in the decomposition (7.2) and the triangle inequality for (c, θ) near $(1, \theta_0)$, it is then straightforward to show that (a, b) are uniquely defined and satisfy the second bound in (7.4). □

By Lemma 7.1, any global solution $\alpha(t) \in h^1$ of the conformal flow system (1.1) satisfying for a sufficiently small positive ϵ and for every t,

$$\inf_{\theta \in \mathbb{S}} \|\alpha(t) - e^{i\theta} A(0)\|_{\ell^2} \leq \epsilon \tag{7.5}$$

admits a unique decomposition in the form

$$\alpha_n(t) = e^{i\theta(t)}(c(t)A_n(0) + a_n(t) + ib_n(t)), \tag{7.6}$$

where the remainder terms satisfy the symplectic orthogonality conditions

$$\langle MA(0), a(t)\rangle = \langle MA(0), b(t)\rangle = 0. \tag{7.7}$$

Now we shall apply this decomposition to control the global solution starting from a small perturbation of the $N = 0$ single-mode state.

LEMMA 7.2. *Assume that initial data $\alpha(0) \in h^1$ satisfy*

$$\|\alpha(0) - A(0)\|_{h^1} \le \delta \tag{7.8}$$

for some sufficiently small $\delta > 0$. Then, the corresponding unique global solution $\alpha(t) \in C(\mathbb{R}, h^1)$ of (1.1) *can be represented by the decomposition* (7.6)–(7.7) *satisfying for all $t \in \mathbb{R}$:*

$$|c(t) - 1| \lesssim \delta, \quad \|a(t) + ib(t)\|_{h^1} \lesssim \delta^{1/2}. \tag{7.9}$$

PROOF. Since $A_n(0) = \delta_{n0}$, the orthogonality conditions (7.7) yield $a_0 = b_0 = 0$. Substituting the representation (7.6) into the conservation laws (1.7) and (1.8), we obtain

$$Q(\alpha(0)) = Q(\alpha(t)) = c(t)^2 + \sum_{n=1}^{\infty}(n+1)\big(a_n^2 + b_n^2\big), \tag{7.10}$$

and

$$E(\alpha(0)) = E(\alpha(t)) = c(t)^2 + \sum_{n=1}^{\infty}(n+1)^2\big(a_n^2 + b_n^2\big). \tag{7.11}$$

Thanks to (7.8) we have

$$|Q(\alpha(0)) - 1| \lesssim \delta, \quad |E(\alpha(0)) - 1| \lesssim \delta. \tag{7.12}$$

Subtracting (7.10) from (7.11) and using (7.12), we obtain

$$\sum_{n=1}^{\infty} n(n+1)\big(a_n^2 + b_n^2\big) \lesssim \delta,$$

which yields the second bound in (7.9). Substituting this bound into (7.11) we obtain the first bound in (7.9). By continuity of the solution $\alpha(t) \in C(\mathbb{R}, h^1)$ of (1.1), the bound (7.5) is satisfied for every $t \in \mathbb{R}$ if it is satisfied for $t = 0$. Therefore, by Lemma 7.1, the decomposition (7.6)–(7.7) holds for every t and the bounds (7.9) are continued for every t. $\square$

Theorem 1.3 is the reinstatement of the result of Lemma 7.2 with $\epsilon = \mathcal{O}(\delta^{1/2})$ or, equivalently, $\delta = \mathcal{O}(\epsilon^2)$.

Remark 7.3. Bounds in (7.9) can be improved for perturbations with $\alpha_0(0) = 1$ since then $c(0) = 1$, $a_0(0) = b_0(0) = 0$ in the decomposition (7.6). In this case, δ is replaced by δ^2 in (7.12); hence $\delta^{1/2}$ is replaced by δ in the bounds (7.9).

8 Orbital Stability of the Ground State

Here we prove Theorem 1.4. As the first step we establish some coercivity estimates for the operators $L_\pm(p)$ defined in (6.1) and (6.2). Let us redefine $[X_c(p)]^\perp$ in (6.24) by using a different but equivalent choice of the two orthogonality conditions:

$$[X_c(p)]^\perp := \{a \in \ell^2(\mathbb{N})\colon \langle MA(p), a\rangle = \langle MA'(p), a\rangle = 0\}. \tag{8.1}$$

This is a *symplectically orthogonal* subspace to $X_0(p) = \text{span}\{A(p), A'(p)\}$, the two-dimensional subspace associated with the positive and zero eigenvalues of $L_\pm(p)$ (recall (6.10), which relates $A'(p)$ to $A(p)$ and $MA(p)$). Let us introduce the symplectically orthogonal projection operator $\Pi_c(p)$: $\ell^2(\mathbb{N}) \to [X_c(p)]^\perp \subset \ell^2(\mathbb{N})$. The following lemma shows that the operators $\Pi_c(p)L_\pm(p)\Pi_c(p)$ are negative and coercive on $[X_c(p)]^\perp$ and that the coercivity constant is independent of $p \in [0,1)$.

LEMMA 8.1. *Given $a \in h^{1/2}$, for every $p \in [0,1)$ we have*

$$\langle \Pi_c(p)L_\pm(p)\Pi_c(p)a, a\rangle \lesssim -\|a\|^2_{h^{1/2}}. \tag{8.2}$$

PROOF. Recall that $L_+(p)$ has the simple positive eigenvalue $\lambda_*(p)$ with the eigenvector $MA(p)$ and the simple zero eigenvalue with the eigenvector $A'(p)$, whereas the remaining eigenvalues are negative. Using (6.11), we have

$$\langle [L_+(p)]^{-1}MA(p), MA(p)\rangle = \frac{1}{2}\langle A(p), MA(p)\rangle > 0. \tag{8.3}$$

By theorem 4.1 in [10], this implies that the positive eigenvalue of $L_+(p)$ becomes a strictly negative eigenvalue of $\mathcal{L}_+(p) := \Pi_c(p)L_+(p)\Pi_c(p)$ under the first constraint in (8.1). On the other hand, since

$$\langle MA'(p), A'(p)\rangle > 0, \tag{8.4}$$

the zero eigenvalue of $L_+(p)$ becomes a strictly negative eigenvalue of $\mathcal{L}_+(p)$ under the second constraint in (8.1). Thus, $\mathcal{L}_+(p)$ is strictly negative with the spectral gap (the distance between the negative spectrum of $\mathcal{L}_+(p)$ and zero). The coercivity bound (8.2) for $\mathcal{L}_+(p)$ follows from standard spectral theorem and the Gårding inequality since the quadratic form for $\mathcal{L}_+(p)$ is bounded in $h^{1/2}(\mathbb{N})$.

The operator $L_-(p)$ has a double zero eigenvalue and the remaining eigenvalues are negative. The eigenvectors for the double zero eigenvalue coincide with $MA(p)$ and $A'(p)$, thanks to (6.10), which relates $A'(p)$ to $A(p)$ and $MA(p)$. The same argument as above yields the bound (8.2) for $\mathcal{L}_-(p) := \Pi_c(p)L_-(p)\Pi_c(p)$. □

In the second step we decompose a solution of the system (1.1) into a four-parameter family of ground states generated by the scaling (1.4) and gauge symmetries (1.5) and (1.6), the parameter $p \in [0,1)$, as well as the symplectically orthogonal remainder term. More precisely, we have:

LEMMA 8.2. *For every $p_0 \in (0,1)$, there exists $\delta_0 > 0$ such that for every $\alpha \in \ell^2$ satisfying*

$$\delta := \inf_{\theta,\mu\in\mathbb{S}} \|\alpha - e^{i(\theta+\mu+\mu\cdot)}A(p_0)\|_{\ell^2} \le \delta_0, \tag{8.5}$$

there exists a unique choice for real-valued numbers (c, p, θ, μ) and real-valued sequences $a, b \in \ell^2$ in the orthogonal decomposition

$$\alpha_n = e^{i(\theta+\mu+\mu n)}(cA_n(p) + a_n + ib_n), \tag{8.6}$$

subject to the orthogonality conditions

$$\langle MA(p), a\rangle = \langle MA'(p), a\rangle = \langle MA(p), b\rangle = \langle MA'(p), b\rangle = 0 \tag{8.7}$$

satisfying the estimate

$$|c-1| + |p-p_0| + \|a+ib\|_{\ell^2} \lesssim \delta. \tag{8.8}$$

PROOF. The proof is based on the inverse function theorem applied to the vector function $F(c,p,\theta,\mu;\alpha) : \mathbb{R}^4 \times \ell^2 \mapsto \mathbb{R}^4$ given by

$$F(c,p,\theta,\mu;\alpha) := \begin{bmatrix} \langle MA(p), \mathrm{Re}(e^{-i(\theta+\mu+i\mu\cdot)}\alpha - cA(p))\rangle \\ \langle MA'(p), \mathrm{Re}(e^{-i(\theta+i\mu+i\mu\cdot)}\alpha - cA(p))\rangle \\ \langle MA(p), \mathrm{Im}(e^{-i(\theta+i\mu+i\mu\cdot)}\alpha - cA(p))\rangle \\ \langle MA'(p), \mathrm{Im}(e^{-i(\theta+i\mu+i\mu\cdot)}\alpha - cA(p))\rangle \end{bmatrix},$$

The Jacobian matrix DF at $\alpha = A(p_0)$ is block diagonal

$$DF(1,p_0,0,0;A(p_0)) = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix},$$

where

$$D_1 = -\begin{bmatrix} \langle MA(p_0), A(p_0)\rangle & 0 \\ \langle MA'(p_0), A(p_0)\rangle & \langle MA'(p_0), A'(p_0)\rangle \end{bmatrix}$$

and

$$D_2 = -\begin{bmatrix} \langle MA(p_0), A(p_0)\rangle & \langle MA(p_0), MA(p_0)\rangle \\ 0 & \langle MA'(p_0), MA(p_0)\rangle \end{bmatrix}.$$

Therefore, $DF(1,p_0,0,0;A(p_0))$ is invertible with the $\mathcal{O}(1)$ bound on the inverse matrix for every $p_0 \in (0,1)$. Hence, for sufficiently small $\delta > 0$, there exists a unique root (c,p,θ,μ) near $(1,p_0,\theta_0,\mu_0)$, where (θ_0,μ_0) are arguments in the infimum (8.5), with the bound

$$|c-1| + |p-p_0| + |\theta-\theta_0| + |\mu-\mu_0| \lesssim \delta.$$

Thus, the first two bounds in (8.8) are satisfied for c and p. By using the definition of (a,b) in the decomposition (8.6) and the triangle inequality for (c,p,θ,μ) near $(1,p_0,\theta_0,\mu_0)$, it is then straightforward to show that (a,b) are uniquely defined and satisfies the last bound in (8.8). □

For any global solution $\alpha(t) \in C(\mathbb{R}, h^1)$ of (1.1) that stays close to the ground state orbit $\mathcal{A}(p_0)$ in ℓ^2, i.e.,

$$\inf_{\theta,\mu\in\mathbb{S}} \|\alpha(t) - e^{i(\theta+\mu+\mu\cdot)}A(p_0)\|_{\ell^2} \le \epsilon \tag{8.9}$$

for a sufficiently small positive ϵ, Lemma 8.2 yields the unique decomposition in the form

$$\alpha_n(t) = e^{i(\theta(t)+(n+1)\mu(t))}(c(t)A_n(p(t)) + a_n(t) + ib_n(t)), \tag{8.10}$$

where the remainder terms satisfy the symplectic orthogonality conditions

$$\langle MA(p(t)), a(t)\rangle = \langle MA'(p(t)), a(t)\rangle = 0 \tag{8.11}$$

and

$$\langle MA(p(t)), b(t)\rangle = \langle MA'(p(t)), b(t)\rangle = 0. \tag{8.12}$$

By the coercivity bounds in Lemma 8.1, we control $c(t)$, $a(t)$, and $b(t)$ as follows.

LEMMA 8.3. *Assume that the initial data* $\alpha(0) \in h^1$ *satisfy*

$$\|\alpha(0) - A(p_0)\|_{h^1} \leq \delta \tag{8.13}$$

for some $p_0 \in [0,1)$ *and a sufficiently small* $\delta > 0$*. Then, the corresponding unique global solution* $\alpha(t) \in C(\mathbb{R}, h^1)$ *of* (1.1) *can be represented by the decomposition* (8.10) *with* (8.11) *and* (8.12) *satisfying for all* t:

$$|c(t) - 1| + \|a(t) + ib(t)\|_{h^{1/2}} \lesssim \delta. \tag{8.14}$$

PROOF. Let us define the function

$$\Delta(c) := c^2(Q(\alpha) - 1) - \frac{1}{2}(H(\alpha) - 1).$$

Evaluating this function for the decomposition (8.10) with (8.11) and (8.12), and using the variational characterization of the ground state (1.12), we obtain

$$\begin{aligned}
\Delta(c(t)) &= c^2(t)(Q(\alpha(t)) - 1) - \frac{1}{2}(H(\alpha(t)) - 1) \\
&= \frac{1}{2}\big(c(t)^2 - 1\big)^2 - c(t)^2\langle L_+(p)a(t), a(t)\rangle \\
&\quad - c(t)^2\langle L_-(p)b(t), b(t)\rangle + N(a(t), b(t), c(t)),
\end{aligned} \tag{8.15}$$

where the cubic and quartic terms in N satisfy the bound

$$|N(a,b,c)| \lesssim |c|\|a + ib\|^3_{h^{1/2}} + \|a + ib\|^4_{h^{1/2}}. \tag{8.16}$$

On the other hand, we have

$$\Delta(c(t)) = \Delta(1) + (c(t)^2 - 1)(Q(\alpha(t)) - 1), \tag{8.17}$$

where, thanks to (8.13) and the conservation of $Q(\alpha(t))$ and $H(\alpha(t))$, we have

$$|\Delta(1)| \lesssim \delta^2, \quad |Q(\alpha(0)) - 1| \lesssim \delta. \tag{8.18}$$

Combining (8.15) with (8.17), using the bounds (8.16) and (8.18), as well as the coercivity bounds (8.2) in Lemma 8.1, we obtain

$$(c(t)^2 - 1)^2 + \|a(t)\|^2_{h^{1/2}} + \|b(t)\|^2_{h^{1/2}} \lesssim \delta^2,$$

which yields the proof of the bound (8.14). □

The bound (8.14) controls the drift of the parameter $c(t)$ and the perturbation terms $a(t)$ and $b(t)$ in the $h^{1/2}$-norm. The following lemma uses the conservation of $E(\alpha)$ and $Z(\alpha)$ to control the drift of the parameter $p(t)$ and the perturbation terms $a(t)$ and $b(t)$ in the h^1-norm.

LEMMA 8.4. *Under the same assumption* (8.13) *for the initial data* $\alpha(0)$ *as in Lemma* 8.3*, the corresponding solution* $\alpha(t) \in C(\mathbb{R}, h^1)$ *of* (1.1) *can be represented by the decomposition* (8.10) *with* (8.11) *and* (8.12) *satisfying for all* t:

$$|p(t) - p_0| \lesssim \delta, \quad \|a(t) + ib(t)\|_{h^1} \lesssim \delta^{1/2}. \tag{8.19}$$

PROOF. Substituting the decomposition (8.10) into the expression for $E(\alpha)$ and using the orthogonality conditions (8.11) and (8.12), we obtain

$$E(\alpha(t)) = \frac{1 + p(t)^2}{1 - p(t)^2} c(t)^2 + \langle Ma(t), Ma(t)\rangle + \langle Mb(t), Mb(t)\rangle. \tag{8.20}$$

Thanks to (8.13) and the conservation of $E(\alpha(t))$, we have $E(\alpha(t)) = E(\alpha(0))$ with

$$\left| E(\alpha(0)) - \frac{1 + p_0^2}{1 - p_0^2} \right| \lesssim \delta. \tag{8.21}$$

Combining (8.20) and (8.21) and using $|c(t) - 1| \lesssim \delta$ from the bound (8.14), we obtain

$$\frac{2(p(t)^2 - p_0^2)}{(1 - p(t)^2)(1 - p_0^2)} + \|a(t) + ib(t)\|_{h^1}^2 \lesssim \delta, \tag{8.22}$$

which eliminates the drift of $p(t)$ towards larger values thanks to the upper bound $p(t) - p_0 \lesssim \delta$.

In order to get the lower bound for the drift of $p(t)$, we substitute the decomposition (8.10) into the expression for $Z(\alpha)$ and obtain

$$Z(\alpha(t)) = e^{-i\mu(t)} \left[\frac{2p(t)}{1 - p(t)^2} c(t)^2 + \sum_{n=0}^{\infty} (n+1)(n+2) \bar{\beta}_{n+1}(t) \beta_n(t) \right], \tag{8.23}$$

where the linear terms vanish thanks to the orthogonality conditions (8.11) and (8.12) and we have denoted $\beta(t) := a(t) + ib(t)$. Since

$$|Z(\alpha(t))| \leq \frac{2p(t)}{1 - p(t)^2} c(t)^2 + \sum_{n=0}^{\infty} (n+1)(n+2) |\beta_{n+1}(t)||\beta_n(t)|,$$

we obtain

$$\begin{aligned} E(\alpha(t)) - |Z(\alpha(t))| &\geq \frac{1 - p(t)}{1 + p(t)} c(t)^2 + \sum_{n=0}^{\infty} (n+1)^2 |\beta_n(t)|^2 \\ &\quad - \sum_{n=0}^{\infty} (n+1)(n+2) |\beta_{n+1}(t)||\beta_n(t)| \\ &= \frac{1 - p(t)}{1 + p(t)} c(t)^2 + \frac{1}{2} |\beta_0(t)|^2 \\ &\quad + \frac{1}{2} \sum_{n=0}^{\infty} [(n+1)|\beta_n(t)| - (n+2)|\beta_{n+1}(t)|]^2, \end{aligned}$$

which yields

$$E(\alpha(t)) - |Z(\alpha(t))| \geq \frac{1-p(t)}{1+p(t)} c(t)^2. \tag{8.24}$$

Thanks to (8.13) and the conservation of $E(\alpha(t))$ and $Z(\alpha(t))$, we have

$$E(\alpha(t)) - |Z(\alpha(t))| = E(\alpha(0)) - |Z(\alpha(0))| \tag{8.25}$$

with

$$\left| E(\alpha(0)) - |Z(\alpha(0))| - \frac{1-p_0}{1+p_0} \right| \lesssim \delta. \tag{8.26}$$

Combining (8.24), (8.25), and (8.26) and using $|c(t)-1| \lesssim \delta$ from the bound (8.14), we obtain

$$\frac{p_0 - p(t)}{(1+p(t))(1+p_0)} \lesssim \delta, \tag{8.27}$$

which eliminates the drift of $p(t)$ towards smaller values thanks to the lower bound $p_0 - p(t) \lesssim \delta$. Bounds (8.22) and (8.27) yield (8.19). □

Theorem 1.4 is a reinstatement of the results of Lemmas 8.3 and 8.4 with $\epsilon = \mathcal{O}(\delta^{1/2})$ or, equivalently, $\delta = \mathcal{O}(\epsilon^2)$.

Appendix A Conservation of $Z(\alpha)$

Here we prove that the quantity $Z(\alpha)$ in (1.20) is conserved by the flow (1.1). Differentiating $Z(\alpha)$ with respect to time and using (1.1) yields

$$\begin{aligned} i\frac{dZ}{dt} = & \sum_{n+j=k+l} (n+2) S_{njkl} \overline{\alpha}_{n+1} \overline{\alpha}_j \alpha_k \alpha_l \\ & - \sum_{n+j+1=k+l} (n+1) S_{n+1,jkl} \alpha_n \alpha_j \overline{\alpha}_k \overline{\alpha}_l, \end{aligned}$$

where the summation sign denotes triple summations with respect to (n, j, k) with the constraint on l. By shifting the index n to $n-1$ in the first sum, relabeling the indices $(n \leftrightarrow k, j \leftrightarrow l)$ in the second sum, and employing the symmetries of S_{njkl}, we rewrite the result as

$$i\frac{dZ}{dt} = \sum_{n+j=k+l+1} \left[(n+1) S_{n-1,jkl} - (k+1) S_{nj,k+1,l} \right] \overline{\alpha}_n \overline{\alpha}_j \alpha_k \alpha_l,$$

where we adopt the convention that $S_{njkl} \equiv 0$ whenever any index is negative. It is easy to verify with the explicit expression $S_{njkl} = \min\{n, j, k, l\} + 1$ that the symmetric part of the expression in the square bracket

$$\begin{aligned} I_{njkl} = & (n+1) S_{n-1,jkl} + (j+1) S_{n,j-1,kl} \\ & - (k+1) S_{nj,k+1,l} - (l+1) S_{njk,l+1} \end{aligned}$$

vanishes for every $n + j = k + l + 1$, hence $\frac{dZ}{dt} = 0$. We remark that the identity $I_{njkl} = 0$ is a special case of a general identity that ensures the conservation of a quantity analogous to Z for general cubic resonant flows [2].

Appendix B Some Useful Identities

Let us record identities for partial sums of geometric series:

$$\sum_{k=0}^{n} p^{2k} = \frac{1 - p^{2n+2}}{1 - p^2}, \tag{B.1}$$

$$\sum_{k=0}^{n} k p^{2k} = \frac{p^2(1 - (n+1)p^{2n} + np^{2n+2})}{(1 - p^2)^2}, \tag{B.2}$$

$$\begin{aligned} \sum_{k=0}^{n} k^2 p^{2k} &= \frac{p^2\big(1 + p^2 - (n+1)^2 p^{2n}\big)}{(1 - p^2)^3} \\ &\quad + \frac{p^2\big((2n^2 + 2n - 1)p^{2n+2} - n^2 p^{2n+4}\big)}{(1 - p^2)^3}. \end{aligned} \tag{B.3}$$

In order to verify the expression (6.2), we have used the following computations:

$$\begin{aligned} j \le n\colon \quad & \sum_{k=0}^{\infty} S_{njk,n+k-j}\, p^{n+2k-j} \\ &= \sum_{k=0}^{j} (k+1) p^{n+2k-j} + \sum_{k=j+1}^{\infty} (j+1) p^{n+2k-j} \\ &= \frac{1}{(1 - p^2)^2} (p^{n-j} - p^{2+j+n}), \end{aligned} \tag{B.4}$$

$$\begin{aligned} j \ge n\colon \quad & \sum_{k=j-n}^{\infty} S_{njk,n+k-j}\, p^{n+2k-j} \\ &= \sum_{k=j-n}^{j} (n+k-j+1) p^{n+2k-j} + \sum_{k=j+1}^{\infty} (n+1) p^{n+2k-j} \\ &= \frac{1}{(1 - p^2)^2} (p^{j-n} - p^{2+j+n}), \end{aligned} \tag{B.5}$$

$$
\begin{aligned}
j \geq n:\ & \sum_{k=0}^{n+j} S_{njk,n+j-k} \\
&= \sum_{k=0}^{n}(k+1) + \sum_{k=n+1}^{j}(n+1) + \sum_{k=j+1}^{j+n}(n+j-k+1) \qquad \text{(B.6)} \\
&= (1+j)(1+n),
\end{aligned}
$$

where the identities (B.1)–(B.3) have been used.

In order to verify the expression (6.16) for the eigenvector to the eigenvalue -1 of the operators $L_{\pm}(p)$, or equivalently, of the operator $2T(p) - M$, we write explicitly

$$
\begin{aligned}
\left[(2T(p) - M)v^{(1)}\right]_n = p^2\left[2p^n - \delta_{n0}\right] + (1-p^2)\sum_{k=1}^{\infty}\left[2p^{|n-k|} - (n+1)\delta_{nk}\right] \\
\times \left[k(1-p^2) - (1+p^2)\right]p^{k-2}.
\end{aligned}
$$

By using the identities

$$
\sum_{k=1}^{\infty} p^{k+|n-k|} = \frac{p^2 + n(1-p^2)}{(1-p^2)} p^n, \tag{B.7}
$$

$$
\sum_{k=1}^{\infty} kp^{k+|n-k|} = \frac{2p^2 + n(1-p^4) + n^2(1-p^2)^2}{2(1-p^2)^2} p^n, \tag{B.8}
$$

we have verified that $L_{\pm}v^{(1)} = (2T(p) - M)v^{(1)} = -v^{(1)}$.

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