AN UNUSUAL EIGENVALUE PROBLEM

Piotr Bizoń

M. Smoluchowski Institute of Physics, Jagellonian University
Reymonta 4, 30-059 Cracow, Poland

(Received November 15, 2004)

It is a pleasure to dedicate this work to Professor Andrzej Staruszkiewicz on the occasion of his 65th birthday

We discuss an eigenvalue problem which arises in the studies of asymptotic stability of a self-similar attractor in the sigma model. This problem is rather unusual from the viewpoint of the spectral theory of linear operators and requires special methods to solve it. One of such methods based on continued fractions is presented in detail and applied to determine the eigenvalues.

PACS numbers: 02.30.Jr

1. Introduction

Many nonlinear evolution equations have the property that solutions which are initially smooth become singular after a finite time. The nature of this phenomenon, usually called blowup, has been a subject of intensive studies in many fields ranging from fluid dynamics to general relativity. The problem whether the blowup can occur, and if so, what is its character, is very difficult for some major evolution equations in physics, such as Navier–Stokes equations or Einstein's equations. Thus, in order to get some insight, it seems useful to study toy models. This paper is concerned with such a toy model, a nonlinear radial wave equation

$$u_{tt} - u_{rr} - \frac{2}{r}u_r + \frac{\sin(2u)}{r^2} = 0\,, \tag{1}$$

where r is the radial variable and $u = u(t, r)$. Equation (1) describes equivariant wave maps from the $3+1$ dimensional Minkowski spacetime into the three-sphere (see [1] for the derivation). In the physics literature this model is usually referred to as the sigma model.

The central question for equation (1) is whether solutions starting from smooth initial data

$$u(0,r) = f(r), \qquad u_t(0,r) = g(r) \tag{2}$$

may become singular in future? A hint towards answering this question comes from the scaling argument. Note that equation (1) is scale invariant: if $u(t,r)$ is a solution, so is $u_\lambda(t,r) = u(t/\lambda, r/\lambda)$. Under this scaling the conserved energy

$$E[u] = \int_0^\infty \left(u_t^2 + u_r^2 + \frac{2\sin^2 u}{r^2} \right) r^2 dr \tag{3}$$

transforms as a homogenous function of degree one: $E[u_\lambda] = \lambda E[u]$, which means that equation (1) is supercritical in the language of the theory of nonlinear partial differential equation. For supercritical equations it is energetically favorable that solutions shrink to small scales so singularities are expected to develop from sufficiently large (in a suitable norm) initial data. Although there are no rigorous results in this respect for equation (1), an explicit example of a singularity forming from smooth initial data is known. This example, first pointed out by Shatah [2] and later found in closed form by Turok and Spergel [3] is provided by the self-similar solution

$$u(t,r) = U_0(\rho) = 2\arctan(\rho), \quad \text{where} \quad \rho = \frac{r}{T-t} \tag{4}$$

and $T > 0$ is a constant[1]. Since

$$\partial_r U_0(\rho)\Big|_{r=0} = \frac{2}{T-t}\,, \tag{5}$$

the solution $U_0(\rho)$ becomes singular at the center when $t \nearrow T$. By the finite speed of propagation, one can truncate this solution in space to get a smooth solution with compactly supported initial data which blows up in finite time.

In fact, the self-similar solution U_0 is not only an explicit example of singularity formation, but numerical simulations indicate that it appears as an attractor in the dynamics of generic initial data [1]. We conjectured in [1] that generically the asymptotic profile of blowup is universally given by U_0, that is

$$\lim_{t \nearrow T} u(t, (T-t)r) = U_0(r)\,. \tag{6}$$

[1] U_0 is the ground state of a countable family of self-similar solutions U_n ($n = 0, 1, \dots$) [4]. However, all $n > 0$ solutions are unstable so they do not appear in the dynamics for generic initial data.

To prove this conjecture one needs to understand the mechanism responsible for the process of local convergence to the self-similar solution U_0. Such a mechanism is relatively well-understood for nonlinear diffusion equations where the global dissipation of energy is responsible for the convergence to an attractor, however very little is known for conservative wave equations where the local dissipation of energy is due to dispersion.

In this paper, as the first step towards proving the above conjecture, we describe in more detail how the limit (6) is attained. To this end, in Section 2 we consider the problem of linear stability of the solution U_0. This leads to an eigenvalue problem which is rather unusual from the standpoint of spectral theory of linear operators. In Section 3 we solve this problem using the method of continued fractions. Finally, in Section 4 we present the numerical evidence that the deviation of the dynamical solution from the self-similar attractor is asymptotically well described by the least damped eigenmode.

2. Formulation of the eigenvalue problem

In order to analyze the problem of linear stability of the self-similar solution U_0 it is convenient to define the new time coordinate $\tau = -\ln(T-t)$ and rewrite equation (1) in terms of $U(\tau,\rho) = u(t,r)$

$$U_{\tau\tau} + U_\tau + 2\rho\, U_{\rho\tau} - (1-\rho^2)(U_{\rho\rho} + \frac{2}{\rho}U_\rho) + \frac{\sin(2U)}{\rho^2} = 0\,. \tag{7}$$

In these variables the problem of finite time blowup in converted into the problem of asymptotic convergence for $\tau \to \infty$ towards the stationary solution $U_0(\rho)$. Following the standard procedure we seek solutions of equation (7) in the form $U(\tau,\rho) = U_0(\rho) + w(\tau,\rho)$. Neglecting the $O(w^2)$ terms we obtain a linear evolution equation for the perturbation $w(\tau,\rho)$

$$w_{\tau\tau} + w_\tau + 2\rho\, w_{\rho\tau} - (1-\rho^2)(w_{\rho\rho} + \frac{2}{\rho}w_\rho) + \frac{2\cos(2U_0)}{\rho^2}w = 0\,. \tag{8}$$

Substituting $w(\tau,\rho) = e^{\lambda\tau}v(\rho)/\rho$ into (8) we get the eigenvalue equation

$$-(1-\rho^2)v'' + 2\lambda\rho v' + \lambda(\lambda-1)v + \frac{V(\rho)}{\rho^2}v = 0\,, \tag{9}$$

where

$$V(\rho) = 2\cos(4\arctan\rho) = \frac{2(1-6\rho^2+\rho^4)}{(1+\rho^2)^2}\,. \tag{10}$$

We consider equation (9) on the interval $0 \le \rho \le 1$, which corresponds to the interior of the past light cone of the blowup point $(t=T, r=0)$. Since a solution of the initial value problem for equation (1) starting from smooth initial data remains smooth for all times $t < T$, we demand the solution $v(\rho)$ to be analytic at the both endpoints $\rho = 0$ (the center) and $\rho = 1$ (the past light cone). Such a globally analytic solution of the singular boundary value problem can exist only for discrete values of the parameter λ, hereafter called eigenvalues.

A straightforward way to find the eigenvalues would be to use the Frobenius method. The indicial exponents at the regular singular point $\rho = 0$ are 2 and -1, hence the solution which is analytic at $\rho = 0$ has the power series representation

$$v_0(\rho) = \sum_{n=0}^{\infty} \alpha_n \rho^{2n+2}\,, \qquad \alpha_0 \neq 0\,. \tag{11}$$

Since there are no complex singularities in the open disk of radius 1 about $\rho = 0$, the series (11) is absolutely convergent for $0 \leq \rho < 1$. At the second regular singular point, $\rho = 1$, the indicial exponents are 0 and $1 - \lambda$ so, as long as λ is not an integer (below we shall discuss this case separately), the two linearly independent solutions have the power series representations

$$v_1(\rho) = \sum_{n=0}^{\infty} \beta_n^{(1)} (1-\rho)^n, \qquad v_2(\rho) = \sum_{n=0}^{\infty} \beta_n^{(2)} (1-\rho)^{n+1-\lambda}\,. \tag{12}$$

These series are absolutely convergent for $0 < \rho \leq 1$. If λ is not an integer then only the solution $v_1(\rho)$ is analytic at $\rho = 1$. From the theory of linear ordinary differential equations we know that the three solutions $v_0(\rho)$, $v_1(\rho)$, and $v_2(\rho)$ are connected on the interval $0 < \rho < 1$ by the linear relation

$$v_0(\rho) = A(\lambda) v_1(\rho) + B(\lambda) v_2(\rho)\,. \tag{13}$$

The requirement that the solution which is analytic at $\rho = 0$ is also analytic at $\rho = 1$ serves as the quantization condition for the eigenvalues $B(\lambda) = 0$. Unfortunately, the explicit expressions for the connection coefficients $A(\lambda)$ and $B(\lambda)$ are not known for equations with more than three regular singular points. In the next section we shall present an indirect method which goes around this difficulty.

We remark in passing that alternatively the eigenvalues can be computed numerically using a shooting-to-a-midpoint technique. With this technique one approximates the solutions $v_0(\rho)$ and $v_1(\rho)$ by the power series (11) and (12), truncated at some sufficiently large n, and then computes the Wronskian of these solutions at a midpoint, $\rho = 1/2$, say. The zeros of the Wronskian correspond to the eigenvalues. Although this technique generates the eigenvalues with reasonable accuracy, it is computationally very costly, especially for large negative values of λ, because the power series (11) and

(12) converge very slowly. Note also that shooting towards $\rho = 1$ fails completely for large negative λ because the solution $v_2(\rho)$ is subdominant at $\rho = 1$, that is, it is negligible with respect to the analytic solution $v_1(\rho)$.

3. Solution of the eigenvalue problem

In this section we shall solve the eigenvalue problem (9) using a method which exploits an intimate relationship between recurrence relations and continued fractions. Although this method is not widely known, it is in fact quite old and has been applied in the past to determine the bound states of the hydrogen molecule ion [5] and quasinormal modes of black holes [6].

The key idea is to determine the analyticity properties of the power series solution $v_0(\rho)$ from the asymptotic behavior of the expansion coefficients α_n. In order to implement this idea it is convenient to change the variables

$$v(\rho) = (2-x)^{\frac{\lambda-1}{2}} y(x), \qquad x = \frac{2\rho^2}{1+\rho^2} \,. \tag{14}$$

In terms of these variables equation (9) takes the form

$$x^2(1-x)(2-x)y'' + x[1-(1+\lambda)x(2-x)]y' - \tfrac{1}{4}[\lambda^2 x(1-x) + 9x^2 - 17x + 4]y = 0 \,. \tag{15}$$

The reason of making the transformation (14) is twofold. First, the transformation of the independent variable rearranges the singular points of equation (9) in such a way that, without changing the points $\rho = 0$ and $\rho = 1$, moves the bothersome singularities at $\rho = \pm i$ (which lie on the unit disk around $\rho = 0$ and obstruct the analysis of analyticity of the power series (11) at $\rho = 1$) to infinity and moves $\rho = \infty$ to $x = 2$. Second, by factoring out the singular behavior at $x = 2$ the number of terms in the recurrence relation for the coefficients of the power series solution around $x = 0$ is reduced from four to three. The indicial exponents at $x = 0$ are 1 and $-1/2$ so the solution which is analytic at $x = 0$ has a power series expansion

$$y_0(x) = \sum_{n=1}^{\infty} a_n x^n, \quad a_1 \neq 0 \,. \tag{16}$$

Substituting this series into equation (15) we get the three-term recurrence relation

$$\begin{aligned} &p_2(0)a_2 + p_1(0)a_1 = 0, \\ &p_2(n)a_{n+2} + p_1(n)a_{n+1} + p_0(n)a_n = 0 \,, \quad n = 1, 2, \ldots \end{aligned} \tag{17}$$

with the initial conditions $a_0 = 0$ and $a_1 = 1$ (normalization) where

$$\begin{aligned} p_2(n) &= 8n^2 + 28n + 20\,, && (18)\\ p_1(n) &= -12n^2 - (20+8\lambda)n - \lambda^2 - 8\lambda + 9\,, && (19)\\ p_0(n) &= 4n^2 + 4\lambda n + \lambda^2 - 9\,. && (20) \end{aligned}$$

The series (16) is absolutely convergent for $0 \leq x < 1$ and in general is divergent for $x > 1$. In order to determine the analyticity properties of the solution $y_0(x)$ at $x = 1$ we need to find the large n behavior of the expansion coefficients a_n. The three-term recurrence relation (17) can be viewed as the second order difference equation so it has two linearly independent asymptotic solutions for $n \to \infty$. Following standard methods (see, for example, [7]) we find

$$a_n^{(1)} \sim n^{\lambda-2} \sum_{s=0}^{\infty} \frac{c_s^{(1)}}{n^s}, \quad \text{and} \quad a_n^{(2)} \sim 2^{-n} n^{-\frac{3}{2}} \sum_{s=0}^{\infty} \frac{c_s^{(2)}}{n^s}\,, \tag{21}$$

where the coefficients $c_s^{(1,2)}$ can be determined recursively from (17). Thus, in general the solution of the recurrence relation (17) behaves asymptotically as

$$a_n \sim c_1(\lambda) a_n^{(1)} + c_2(\lambda) a_n^{(2)}\,. \tag{22}$$

If the coefficient $c_1(\lambda)$ is nonzero then

$$\frac{a_{n+1}}{a_n} \sim \frac{a_{n+1}^{(1)}}{a_n^{(1)}} \to 1 \quad \text{as} \quad n \to \infty\,, \tag{23}$$

hence the power series (16) is divergent for $x > 1$ (in fact, it has a branch point singularity at $x = 1$). On the other hand, if $c_1(\lambda) = 0$ then

$$\frac{a_{n+1}}{a_n} \sim \frac{a_{n+1}^{(2)}}{a_n^{(2)}} \to \frac{1}{2} \quad \text{as} \quad n \to \infty\,, \tag{24}$$

and the power series (16) is absolutely convergent for $x < 2$, in particular the solution $y_0(x)$ is analytic at $x = 1$.

The advantage of replacing the quantization condition $B(\lambda) = 0$ in the connection formula (13) by the equivalent condition $c_1(\lambda) = 0$ follows from the fact that $c_1(\lambda)$ is the coefficient of the dominant solution in (22), in contrast to $B(\lambda)$ which is the coefficient of the subdominant solution in (13). In the theory of difference equations (recurrence relations) a subdominant solution, that is a solution which is asymptotically negligible with respect to any other solution, is called the minimal solution. In contrast to a dominant solution, the minimal solution, if it exists, is unique. The condition $c_1(\lambda) = 0$ is thus equivalent to the requirement that the solution of the recurrence relation (17) starting with $a_0 = 0$ and $a_1 = 1$ is minimal. In order to find when this minimal solution exists we shall use now a relationship between three-term recurrence relations and continued fractions. Let

$$A_n = \frac{p_1(n)}{p_2(n)}, \quad B_n = \frac{p_0(n)}{p_2(n)}, \quad r_n = \frac{a_{n+1}}{a_n}. \tag{25}$$

Then, we can rewrite (17) as

$$r_n = -\frac{B_n}{A_n + r_{n+1}}, \tag{26}$$

and applying this formula repeatedly we get the continued fraction representation of r_n

$$r_n = -\frac{B_n}{A_n -}\,\frac{B_{n+1}}{A_{n+1} -}\,\frac{B_{n+2}}{A_{n+2} -}\cdots \tag{27}$$

Pincherle's theorem [7] says that the continued fraction on the right hand side of equation (27) converges if and only if the recurrence relation (17) has a minimal solution $a_n^{\min}$ and, moreover, in the case of convergence, equation (27) holds with $r_n = a_{n+1}^{\min}/a_n^{\min}$ for each n.

Using Pincherle's theorem and setting $n = 1$ in (27) we obtain the eigenvalue equation

$$\frac{a_2}{a_1} = \frac{1}{20}(\lambda^2 + 8\lambda - 9) = -\frac{B_1(\lambda)}{A_1(\lambda) -}\,\frac{B_2(\lambda)}{A_2(\lambda) -}\cdots \tag{28}$$

The continued fraction in (28), which by Pincherle's theorem is convergent for any λ, can be approximated with essentially arbitrary accuracy by downward recursion starting from a sufficiently large $n = N$ and some (arbitrary) initial value r_N. The roots of the transcendental equation (28) are then found numerically (see Table I).

TABLE I

The first twelve eigenvalues.

n	0	1	2	3	4	5
λ_n	1	–0.542466	–2	– 3.398382	–4.765079	–6.102295
n	6	7	8	9	10	11
λ_n	–7.297807	–7.765347	–8.853889	–10.1228208	–11.196495	–11.802614

A glance at Table I shows some interesting properties of the spectrum. First, all the eigenvalues are real. Second, there are no eigenvalues $\lambda > 1$. Third, there are two integer eigenvalues $\lambda = 1$ and $\lambda = -2$. Below we discuss these properties in detail.

Why the eigenvalues are real?

We find this property surprising because we cannot see any a priori reason which forbids complex eigenvalues. On the contrary, it is easy to deform the potential V in equation (9), without changing the character of singularities at the endpoints, in such a way that the resulting eigenvalues are complex. Thus, the reality of eigenvalues is related to the special form of the potential V. It remains a puzzle whether this relationship is accidental or there is something deep to it.

The eigenvalue $\lambda = 1$

This eigenvalue is due to the freedom of changing the blowup time T. To see this, consider a solution $U_0(\frac{r}{T'-t})$ with shifted blowup time. In terms of the similarity variables $\tau = -\ln(T-t)$ and $\rho = r/(T-t)$ we have

$$U_0\left(\frac{r}{T'-t}\right) = U_0\left(\frac{\rho}{1+\epsilon e^{\tau}}\right), \quad \text{where} \quad \epsilon = T' - T\,. \tag{29}$$

Thus, the perturbation induced by the shift of blowup time has the form

$$w(\rho,\tau) = -\epsilon e^{\tau} \rho U_0'(\rho)\,, \tag{30}$$

and consequently the mode

$$v(\rho) = \rho^2 U_0'(\rho) = \frac{\rho^2}{1+\rho^2} \tag{31}$$

solves equation (9) with $\lambda = 1$. Since this mode is evidently analytic at both $\rho = 0$ and $\rho = 1$, we conclude that $\lambda = 1$ is an eigenvalue. We emphasize that this positive eigenvalue should not be interpreted as the physical instability of the solution U_0, as it is an artifact of introducing the similarity variables and does not show up in the dynamics for $u(t,r)$.

Nonexistence of eigenvalues $\lambda > 1$

Using the transformation

$$v(\rho) = (1-\rho^2)^{-\frac{\lambda}{2}} z(\rho)\,, \tag{32}$$

we can put equation (9) into a standard Sturm–Liouville form

$$Az = \mu z, \quad \text{where} \quad A = -(1-\rho^2)^2 \frac{d^2}{d\rho^2} + \frac{1-\rho^2}{\rho^2} V(\rho)\,, \quad \mu = \lambda(2-\lambda). \tag{33}$$

In the space of functions

$$\mathcal{D} = L^2\left([0,1], \frac{d\rho}{(1-\rho^2)^2}\right)$$

the operator A is self-adjoint so the eigenvalues μ are real. Both endpoints are of the limit-point type. Near $\rho = 0$ an admissible (that is, belonging to $\mathcal{D}$) solution behaves as $z(\rho) \sim \rho$. Near $\rho = 1$ two independent solutions behave as

$$z_{\pm}(\rho) \sim (1-\rho)^{\frac{1}{2}(1\pm\sqrt{1-\mu})}, \tag{34}$$

so only the solution z_+ with $\mu < 1$ is admissible. Now, we shall show the operator A has no eigenvalues. To see this, note that the solution with $\mu = 1$, corresponding to the gauge mode

$$z(\rho) = \sqrt{1-\rho^2}\frac{\rho^2}{1+\rho^2}, \tag{35}$$

has no zeros. This implies by a standard theorem for Sturm–Liouville operators that there are no eigenvalues below $\mu = 1$. Since $\mu = 1$ is not an eigenvalue (because the mode (35) does not belong to $\mathcal{D}$), we conclude that the operator A has the purely continuous spectrum $\mu \geq 1$.

What does this fact tell us about the eigenvalues of our problem? Rewriting (34) in terms of λ we have

$$z_+(\rho) \sim (1-\rho)^{\frac{\lambda}{2}}, \quad z_-(\rho) \sim (1-\rho)^{1-\frac{\lambda}{2}}, \tag{36}$$

so comparing with (32) we see that only z_+ leads to a solution $v(\rho)$ which is analytic at $\rho = 1$. For $\mathrm{Re}(\lambda) > 1$ the solution z_+ belongs to $\mathcal{D}$ so in this case there is one-to-one relationship between the eigenvalues of the operator A and the eigenvalues of our problem. Since the former has no eigenvalues, we infer that our problem has no eigenvalues with $\mathrm{Re}(\lambda) > 1$. This result was obtained previously in [4].

We point out that for $\mathrm{Re}(\lambda) < 1$ the requirements of square-integrability and analyticity near $\rho = 1$ are mutually exclusive so in this case there is no relationship between the eigenvalues of the operator A and the eigenvalues of our problem.

The algebraically special eigenvalue $\boldsymbol{\lambda = -2}$

If $1-\lambda = N$ is a positive integer then the two linearly independent power series solutions of equation (15) around $x = 1$ are

$$y_1(x) = \sum_{n=0}^{\infty} b_n^{(1)}(1-x)^{n+N}, \qquad y_2(x) = C_N y_1(x)\ln(1-x) + \sum_{n=0}^{\infty} b_n^{(2)}(1-x)^n. \tag{37}$$

The solution $y_1(x)$, corresponding to the larger indicial exponent, is analytic at $x = 1$ but the solution $y_2(x)$, corresponding to the smaller indicial exponent, involves a logarithm in general. However, for $N = 3$ we have an exceptional case: the coefficient C_3 vanishes and both solutions are analytic at $x = 1$. The best way to see this is to solve the recurrence relation for the coefficients $b_n^{(2)}$ assuming temporarily that the solution y_2 contains no logarithm. We find

$$b_n^{(2)} = \frac{P_{2n-1}(\lambda)}{\lambda(\lambda+1)(\lambda+3)...(\lambda+n-1)}\,, \tag{38}$$

where $P_{2n-1}(\lambda)$ is a polynomial in λ of order $2n-1$. Moreover this polynomial has no integer roots. When $1-\lambda = N$ is a positive integer different from 3, then the expansion coefficient $b_N^{(2)}$ does not exist which contradicts the assumption that y_2 contains no logarithm. The only exception is $N = 3$ because the denominator in (38) has no $(\lambda+2)$ term. Thus, for $\lambda = -2$ the singularity at $x = 1$ is apparent - each solution which is analytic at $x = 0$ is automatically analytic at $x = 1$ as well. This proves that $\lambda = -2$ is the eigenvalue.

4. Numerical verification

According to the linear stability analysis presented above the convergence of the solution $u(t, r)$ towards the self-similar attractor U_0 should be described by the formula

$$u(t,r) = U(\tau,\rho) = U_0(\rho) + \sum_{k=1} c_k e^{\lambda_k \tau} v_k(\rho)/\rho \sim U_0(\rho) + c_1 e^{\lambda_1 \tau} v_1(\rho)/\rho$$
$$\text{as} \quad \tau \to \infty\,, \tag{39}$$

where $v_k(\rho)/\rho$ are the eigenmodes corresponding to the eigenvalues λ_k and c_k are the expansion coefficients. In order to verify (39) we solved equation (1) numerically for large initial data leading to blowup, expressed the result in the similarity variables, and computed the deviation from U_0 for $t \nearrow T$. The result (see figure 1) shows that, in perfect agreement with the formula (39), the deviation from U_0 is described by the least damped eigenmode v_1. This makes us feel confident that the calculation presented in Section 3 contains no algebraic errors.

We remark that the formula (39) could be used to compute the eigenvalues numerically directly from the dynamics rather than by solving the eigenvalue equation. Such a computation was performed by Donninger [8] with the result which is in rough agreement with Table I (rough, as the dynamical computation of eigenvalues is by far less accurate than the continued fraction method).

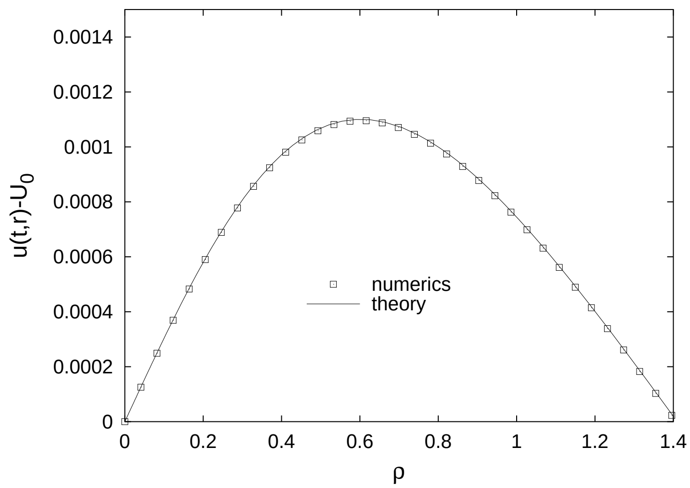

Fig. 1. We plot the deviation of the dynamical solution $u(t, r)$ from the self-similar solution $U_0 = 2\arctan(\rho)$ at some moment of time close to the blowup time. The solid line shows the least damped eigenmode $c_1(T-t)^{-\lambda_1} v_1(\rho)/\rho$, where the coefficient c_1 is fitted once for all times.

I thank Tadek Chmaj for remarks and providing the numerical data for Section 4, and Roland Donninger for informing me about his numerical results. This research was supported in part by the Polish State Committee for Scientific Research (KBN) grant 2 P03B 006 23. The friendly hospitality of the Albert Einstein Institute, where part of this work was done, is greatly appreciated.

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