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HUYGENS' PRINCIPLE AND ANOMALOUSLY SMALL RADIATION TAILS*

Piotr Bizoń

M. Smoluchowski Institute of Physics, Jagellonian University
Reymonta 4, 30-059 Kraków, Poland

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This is a short account of recent joint work with Chmaj and Rostworowski on late-time asymptotic behavior of linear and nonlinear waves propagating on even-dimensional Minkowski spacetime.

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1. Introduction

The work presented in this talk is part of a long-time project aimed at the detailed quantitative description of the process of relaxation to equilibrium for nonlinear wave equations defined on spatially unbounded manifolds. By equilibrium we mean here a stable stationary solution, like a soliton, a black hole, or just a flat space. The convergence to these solutions occurs through a mechanism of radiating an excess energy to infinity. For a large class of physically interesting systems the late stages of this process are universal: for intermediate times the convergence has the form of exponentially damped oscillations (called quasinormal modes) and asymptotically it has the form of polynomial decay (called a tail). This very last stage of the relaxation process, the tail, is the subject of my talk.

The presentation of this talk at the conference devoted to Mathisson is justified by the fact our results touch upon Huygens' principle, one of the main subjects of Mathisson's mathematical interests. Recall that a wave equation is said to satisfy Huygens' principle if: *(i)* the solution at a point P depends only on the initial data at the intersection of the past light cone of P with the Cauchy hypersurface or, equivalently, *(ii)* the solution vanishes at all points which cannot be reached from the initial data by a null geodesic

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(*i.e.*, there is no tail). A prototype equation satisfying Huygens' principle is the ordinary wave equation in $d+1$ dimensional Minkowski spacetime for odd $d \geq 3$. Actually, according to Hadamard's conjecture [1] this is the only (modulo trivial transformations) huygensian linear second-order hyperbolic equation of the form

$$g^{\mu\nu}(x)\nabla_\mu \nabla_\nu \phi + A^\mu(x)\nabla_\mu \phi + B(x)\phi = 0 \,. \tag{1}$$

Mathisson proved this conjecture in the case of four dimensional Minkowski spacetime [2]. Counterexamples to Hadamard's conjecture, which have been found later (see [3] and Roy McLenaghan's talk at this conference), do not change the fact that Huygens' property is a very rare and unstable phenomenon. Thus, it is natural to ask if there are perturbations of the free wave equation which preserve Huygens' property approximately, in the sense that the tail which is left behind the wave front is very small. The existence of such special perturbations in higher even dimensions is a byproduct of our studies of tails.

2. Model and assumptions

We consider equations of the form

$$\Box\phi + V(x)\phi + N(\phi, \nabla\phi, x) = 0\,, \qquad \Box = \partial_t^2 - \Delta\,, \quad (t,x) \in R^{d+1}\,, \tag{2}$$

for spherically symmetric smooth initial data with compact support. Since we want the free part to satisfy Huygens' property, we restrict ourselves to odd spatial dimension $d \geq 3$. Apart from obvious mathematical motivations, there are at least two physical reasons for studying higher dimensions $d > 3$. First, for linear wave equations higher dimensions are equivalent to higher spherical harmonics. This follows from the identity

$$\left(\partial_t^2 - \partial_r^2 - \frac{d-1}{r}\partial_r\right)\phi \;=\; \frac{1}{r^l}\left(\partial_t^2 - \partial_r^2 - \frac{2}{r}\partial_r + \frac{l(l+1)}{r^2}\right)(r^l\phi)\,,$$
$$l \;=\; (d-3)/2\,, \tag{3}$$

which relates the $l=0$ radial wave operator in d space dimensions with the radial wave operator in three space dimensions for the l-th spherical harmonic with $l=(d-3)/2$.

Second, some geometric wave equations in $3+1$ dimensions are equivalent to scalar wave equations in $d+1$ dimensions for $d>3$. For example, equivariant wave maps from R^{3+1} into S^3 satisfy the following equation

$$\left(\partial_t^2 - \partial_r^2 - \frac{2}{r}\partial_r\right)\psi + \frac{\sin(2\psi)}{r^2} = 0\,, \tag{4}$$

which, after substitution $\psi = r\phi$, becomes the nonlinear scalar wave equation in $5+1$ dimensions

$$\left(\partial_t^2 - \partial_r^2 - \frac{4}{r}\partial_r\right)\phi + \frac{4}{3}\phi^3 + \text{higher order terms} = 0\,. \qquad (5)$$

3. Tools

In this section we recall two elementary tools from the theory of linear wave equations. The first tool is a formula for the spherically symmetric solution of the free wave equation $\Box\phi = 0$ in $d+1$ dimensions (hereafter we shall use $l = (d-3)/2$ instead of d):

$$\phi(t,r) = \frac{1}{r^{2l+1}} \sum_{k=0}^{l} \frac{2^{k-l}(2l-k)!}{k!(l-k)!}\, r^k \left(a^{(k)}(t-r) - (-1)^k a^{(k)}(t+r)\right)\,. \qquad (6)$$

This solution, which is a superposition of ingoing and outgoing waves, is parameterized by a single function $a(r)$ uniquely determined by initial data (the superscript in round brackets denotes the k-th derivative).

The second tool is the Duhamel formula for the solution of the inhomogeneous free wave equation $\Box\phi = F(t,r)$ with zero data

$$\begin{aligned} \phi(t,r) &= \frac{1}{2r^{l+1}} \int_0^t d\tau \int_{|t-r-\tau|}^{t+r-\tau} \rho^{l+1} P_l(\mu)\ F(\tau,\rho) d\rho\,, \\ \mu &= \frac{r^2+\rho^2-(t-\tau)^2}{2r\rho}\,, \end{aligned} \qquad (7)$$

where $P_l(\mu)$ is the Legendre polynomial of degree l. This expression can be easily obtained from the standard Green's function formula by integrating out the angular variables [4]. In terms of null coordinates $u = \tau - \rho$ and $v = \tau + \rho$ the Duhamel formula (7) takes a more convenient form

$$\begin{aligned} \phi(t,r) &= \frac{1}{2^{l+3} r^{l+1}} \int_{|t-r|}^{t+r} dv \int_{-v}^{t-r} (v-u)^{l+1} P_l(\mu)\, F(u,v) du\,, \\ \mu &= \frac{r^2+(v-t)(t-u)}{r(v-u)}\,. \end{aligned} \qquad (8)$$

The formulae (6) and (8) will be used repeatedly below.

4. Linear tails

For the clarity of presentation we first consider the linear equation with a potential

$$\Box\phi + \lambda V\phi = 0\,, \qquad (\phi(0,r), \partial_t\phi(0,r)) = (f(r), g(r))\,. \tag{9}$$

The prefactor λ, introduced for convenience, will be assumed small and used as a perturbation parameter. In order to determine the long-time behavior of $\phi(t,r)$ we define the perturbation series

$$\phi = \phi_0 + \lambda\phi_1 + \lambda^2\phi_2 + \dots\,, \tag{10}$$

where ϕ_0 satisfies initial data (9) and all higher ϕ_n have zero data. Substituting this series into equation (9) we get the iterative scheme

$$\Box\phi_0 = 0\,, \quad \Box\phi_1 = -V\phi_0\,, \quad \Box\phi_2 = -V\phi_1\,, \quad etc.\,, \tag{11}$$

which can be solved recursively using the formulae (6) and (8). Assuming that $V(r) \sim r^{-\alpha}$ ($\alpha > 2$) for $r \to \infty$, we showed in [5] that the leading order asymptotic behavior at timelike infinity (fixed r and $t \to \infty$) is given by

$$\phi_1(t,r) = \frac{C(l,\alpha)}{t^{\alpha+2l}}\left[A + \mathcal{O}\left(\frac{1}{t}\right)\right]\,, \tag{12}$$

where[1]

$$C(l,\alpha) = -\frac{2^{\alpha+2l-1}}{(2l+1)!!}\left(\frac{\alpha-3}{2}\right)^{\underline{l}}\left(\frac{\alpha}{2}\right)^{\overline{l}} \qquad \text{and} \qquad A = \int\limits_{-\infty}^{+\infty} a(u)\,du\,. \tag{13}$$

The constant A is the only trace of initial data. The expression (12) was first derived by Ching *et al.* [6] who used Fourier transform methods.

We claim that the first iterate provides a good approximation of the entire tail if λ is sufficiently small, that is

$$\phi(t,r) - \lambda\phi_1(t,r) \sim \mathcal{O}(\lambda^2)t^{-(\alpha+2l)}\,. \tag{14}$$

This basically follows from the fact that all higher-order iterates $\phi_n(t,r)$ decay in the same manner (or faster) as $\phi_1(t,r)$. Of course, the main issue is

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[1] Here we use the notation:

$$x^{\underline{0}} := 1, \qquad x^{\underline{k}} := x\cdot(x-1)\cdot\dots\cdot(x-(k-1)), \quad k>0\,,$$
$$x^{\overline{0}} := 1, \qquad x^{\overline{k}} := x\cdot(x+1)\cdot\dots\cdot(x+(k-1)), \quad k>0\,.$$

whether the perturbation series is convergent; for $l = 0$ (*i.e.*, in three space dimensions) this was proved in [7] but for higher l the problem is open. Note, however, that for practical purposes it is sufficient that the series is asymptotic to the solution.

The numerical verification of (14) shows perfect agreement with analytic predictions [5]. We remark that numerical simulations of tails are not quite trivial even in the radial case because discretization errors generate artificial tails which might mask the true behavior. To eliminate such artifacts one has to use high-order finite difference schemes. In addition, quadruple precision is needed to suppress the accumulation of round-off errors during long-time simulations. For these reasons the simulations of tails are computationally expensive.

5. Nonlinear tails

In this section we consider equation (2) without a potential. For simplicity, we take a pure power nonlinearity with an integer exponent ($p \geq 3$ if $l = 0$ and $p \geq 2$ if $l \geq 1$) (the generalization to other nonlinearities is straightforward)

$$\Box\phi - \phi^p = 0\,, \qquad (\phi(0,r), \partial_t\phi(0,r)) = (\varepsilon f(r), \varepsilon g(r))\,. \tag{15}$$

This time the amplitude of initial data ε plays the role of a small parameter in the perturbation series:

$$\phi = \varepsilon\phi_0 + \varepsilon^2\phi_1 + \varepsilon^3\phi_2 + \dots\,. \tag{16}$$

Substituting (16) into equation (15) we get the iteration scheme

$$\Box\phi_0 = 0\,, \qquad \Box\phi_p = \phi_0^p\,, \qquad etc. \tag{17}$$

As above, we get ϕ_0 using the formula (6) and then evaluate ϕ_p using the Duhamel formula (8). In the limit of timelike infinity we obtain [8]

$$\phi_p(t,r) = \frac{\tilde{C}(l,p)}{t^{(l+1)p-1}}\left[\tilde{A} + \mathcal{O}\left(\frac{1}{t}\right)\right]\,, \tag{18}$$

where

$$\begin{aligned} \tilde{C}(l,p) &= (-1)^l\,\frac{2^{(l+1)(p+1)-1}}{(2l+1)!!}[(l+1)(p-1)-2]^{\underline{l}}\,, \\ \tilde{A} &= \int_{-\infty}^{+\infty} [a^{(l)}(u)]^p\,du\,. \end{aligned} \tag{19}$$

The remarks given above in the linear case apply verbatim to the nonlinear case as well; in particular the perturbation series (16) is known to converge for $l = 0$ [7] and is at least asymptotic to the full solution for $l > 0$.

6. Competition between linear and nonlinear tails

The most interesting situation occurs when both the potential and the nonlinearity are present in equation (2). Then, each of these terms produces its own tail:

$$\text{linear tail} \sim t^{-(\alpha+2l)} \qquad \text{and} \qquad \text{nonlinear tail} \sim t^{1-(l+1)p}\,. \tag{20}$$

Clearly, the tail with slower decay rate is dominant asymptotically, that is

$$\phi(t,r) \sim t^{-\gamma}\,, \qquad \gamma = \min\{\alpha + 2l, (l+1)p - 1\}\,. \tag{21}$$

For $l = 0$ this result (without a coefficient) was first proved by Strauss and Tsutaya [9].

To give an example of the competition of linear and nonlinear tails, let us consider the Skyrme model. Under a spherical symmetry reduction (corotational ansatz), this model reduces to the single nonlinear wave equation for the function $F(t,r)$

$$\partial_t(w\partial_t F) - \partial_r(w\partial_r F) + \sin(2F) + \sin(2F)\left(\frac{\sin^2 F}{r^2} + (\partial_r F)^2 - (\partial_t F)^2\right) = 0\,, \tag{22}$$

where $w = r^2 + 2\sin^2 F$. Regular finite energy solutions of (22) must satisfy the boundary conditions $F(t,0) = 0$ and $F(t,\infty) = m\pi$, where an integer m has the interpretation of the topological degree of the solution. For $m = 1$ equation (22) has a regular static solution $S(r)$ called the skyrmion. This solution is linearly stable and plays the role of a global attractor, that is, every solution starting from smooth finite energy initial data of degree one remains globally regular for all times and asymptotically converges to $S(r)$. The perturbation $\phi(t,r) = \sqrt{w}(F(t,r) - S(r))/r^2$ satisfies the equation

$$\left(\partial_t^2 - \partial_r^2 - \frac{4}{r}\partial_r\right)\phi + V(r)\phi + \frac{4}{3}\phi^3 + \text{higher order terms} = 0\,, \tag{23}$$

where the potential $V(r)$ has no bound states and falls off as r^{-6} for $r \to \infty$. For this equation we have $d = 5\,(l = 1)$, $\alpha = 6$ and $p = 3$, hence from (20)

$$\text{linear tail} \sim t^{-8} \qquad \text{and} \qquad \text{nonlinear tail} \sim t^{-5}\,. \tag{24}$$

Thus, the nonlinear tail is dominant [10]. This example shows that one has to be cautious in drawing conclusions about the asymptotic behavior of solutions of nonlinear wave equations on the basis of linear perturbation analysis — even for small amplitude solutions the nonlinear effects can be dominant.

7. Anomalous tails

An advantage of our approach, in contrast to decay estimates in the form of inequalities, is that we control the coefficient of the leading order term of the tail. This allows us to identify those exceptional cases in which this coefficient vanishes and the decay is faster. We shall refer to such tails as anomalous.

Let us first consider the linear case with the pure inverse power potential near infinity, that is $V(r) = \lambda r^{-\alpha}$ for $r > R$. Then, it follows from (13) that $C(l,\alpha) \propto \left(\frac{\alpha-3}{2}\right)^{\underline{l}} = 0$ if α is an odd integer $\leq 2l+1$, hence there is no tail in the first order. This means that the system

$$\Box\phi_0 = 0\,, \qquad \Box\phi_1 = -V\phi_0\,, \tag{25}$$

is huygensian. In order to find the tail in this exceptional case we need to solve the second iteration equation $\Box\phi_2 = -V\phi_1$ via the Duhamel formula. After a long calculation (which requires the asymptotic expansion of ϕ_1 at null infinity) we get (see [5] for the details)

$$\phi(t,r) \approx \lambda^2\phi_2(t,r) = \lambda^2 \frac{D(l,\alpha)}{t^{2(\alpha+l-1)}}\left[A + \mathcal{O}\left(\frac{1}{t}\right)\right]\,, \tag{26}$$

where the coefficient $D(l,\alpha)$ is given by a complicated but explicit expression (see Eq. (23) in [5]).

Next, consider the pure power nonlinearity $\Box\phi = \phi^p$. It follows from (19) that $\tilde{C}(l,p) \propto [(l+1)(p-1)-2]^{\underline{l}} = 0$ if $p = 2$ and $l \geq 1$. Thus, in higher even dimensions the first order tail vanishes for the quadratic nonlinearity. This implies that the system

$$\Box\phi_0 = 0\,, \qquad \Box\phi_1 = \phi_0^2\,, \tag{27}$$

is huygensian. As before, the leading order behavior of the tail can be obtained by solving the second order equation $\Box\phi_2 = 2\phi_0\phi_1$ via the Duhamel formula. The result (see [8] for the details) is

$$\begin{aligned} \phi(t,r) \;&\approx\; \varepsilon^3\phi_2(t,r) \sim \varepsilon^3 \frac{c(l)}{t^{3l+1}}\,, \\ c \;&=\; (-1)^l \frac{2^{3l}}{2l(2l+1)} \int\limits_{-\infty}^{\infty} a^{(l-1)}(\eta)[a^{(l)}(\eta)]^2 d\eta\,. \end{aligned} \tag{28}$$

Note that quadratic nonlinearities occur frequently in nonlinear perturbation theory so anomalous tails are in fact quite common. As an example, consider the Yang–Mills field in four dimensions with the SO(3) gauge group, so that the potential $A_\alpha(x)$ is the skew-symmetric 3×3 matrix $A^{ij}_\alpha(x)$. For the spherically-symmetric ansatz

$$A^{ij}_\mu(x) = \left(\delta^j_\mu x^i - \delta^i_\mu x^j\right) \phi(t,r)\,, \tag{29}$$

the Yang–Mills equation $\partial_\alpha F^{\alpha\beta} + [A_\alpha, F^{\alpha\beta}] = 0$, where $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta]$, reduces to the scalar semilinear wave equation in $5+1$ dimensions

$$\left(\partial_t^2 - \partial_r^2 - \frac{4}{r}\partial_r\right)\phi + 3\phi^2 + r^2\phi^3 = 0\,. \tag{30}$$

The quadratic term in (30) produces an anomalous tail (28) which is of the same order, $\mathcal{O}(\varepsilon^3)$, as the standard tail (18) produced by the cubic term. Combined together they give (see [11] for the derivation)

$$\begin{aligned} \phi(t,r) &\approx \varepsilon^3 \phi_2(t,r) \sim \varepsilon^3\, c\, t^{-4}\,, \\ c &= -8 \int_{-\infty}^{+\infty} a(u) a'(u)^2\,. \end{aligned} \tag{31}$$

Finally, we remark that although our analysis of tails was restricted to the flat background, many conclusions carry over to more general asymptotically flat spacetimes, in particular black hole spacetimes. For example, applying similar methods one can show that the massless scalar field propagating outside a higher even-dimensional Schwarzschild black hole decays anomalously fast as $\phi \sim t^{-(3d-5)}$ [5]. This suggests that the problem of asymptotic stability of the Schwarzschild black hole is easier in higher dimensions.

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