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Nonexistence of Shrinkers for the Harmonic Map Flow in Higher Dimensions

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We prove that the harmonic map flow from the Euclidean space $\mathbb{R}^d$ into the sphere S^d has no equivariant self-similar shrinking solutions in dimensions $d \geq 7$.

This note is concerned with the harmonic map flow for maps u from the Euclidean space $\mathbb{R}^d$ to the sphere S^d. This flow, defined as the gradient flow for the Dirichlet energy,

$$E(u) = \int_{\mathbb{R}^d} |\nabla u|^2, \tag{1}$$

obeys the nonlinear heat equation

$$u_t = \Delta u + |\nabla u|^2 u, \tag{2}$$

where $(t, x) \in \mathbb{R} \times \mathbb{R}^d$ and $u(t, x) \in S^d \hookrightarrow \mathbb{R}^{d+1}$. This equation is scale invariant: if $u(t, x)$ is a solution, so is $u_\lambda(t, x) = u(t/\lambda^2, x/\lambda)$. Under this scaling $E(u_\lambda) = \lambda^{d-2} E(u)$ which means that $d = 2$ is the critical dimension and higher dimensions are supercritical.

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We consider equivariant maps of the form (where $r = |x|$)

$$u(t, x) = \left(\frac{x}{r} \sin v(t, r), \cos v(t, r)\right). \tag{3}$$

This symmetry assumption reduces equation (2) to the scalar heat equation

$$v_t = v_{rr} + \frac{d-1}{r} v_r - \frac{d-1}{2r^2} \sin(2v). \tag{4}$$

A natural question, important for understanding the global behavior of solutions and formation of singularities, is whether there exist solutions of equation (4) which are invariant under scaling, that is, $v(t/\lambda^2, r/\lambda) = v(t, r)$. Such self-similar solutions come in two kinds: self-similar expanding solutions (expanders for short) of the form

$$v(t, r) = g\left(\frac{r}{\sqrt{t}}\right), \quad t > 0, \tag{5}$$

and self-similar shrinking solutions (shrinkers for short) of the form

$$v(t, r) = f\left(\frac{r}{\sqrt{-t}}\right), \quad t < 0. \tag{6}$$

Expanders are easy to construct in any dimension and well understood (see [2, 6]), so here we will consider only shrinkers. Substituting the ansatz (6) into equation (4) and using the similarity variable $y = r/\sqrt{-t}$, we obtain an ordinary differential equation for $f(y)$ on $y \geq 0$

$$f'' + \left(\frac{d-1}{y} - \frac{y}{2}\right) f' - \frac{d-1}{2y^2} \sin(2f) = 0. \tag{7}$$

It is routine to show that both near the center and near infinity there exist one-parameter families of local smooth solutions satisfying

$$f(0) = 0, \quad f'(0) = a > 0, \tag{8}$$

and

$$f(\infty) = b, \quad \lim_{y\to\infty} y^3 f'(y) = -(d-1)\sin(2b), \tag{9}$$

where a and b are free parameters. The assumption that $a > 0$ is made for later convenience (without loss of generality). The question is: do there exist global smooth solutions satisfying the conditions (8) and (9)? This question has been answered in affirmative for $3 \leq d \leq 6$ by Fan [4]. Using a shooting method, Fan proved that there exists a countable sequence of pairs (a_n, b_n) for which the local solutions satisfying (8) and (9) are smoothly connected by a globally regular solution $f_n(y)$. The positive integer n denotes the number of intersections of the solution $f_n(y)$ with $\pi/2$. More detailed quantitative properties of the shrinkers were studied in [2].

Remark 1. To justify the conditions (8) and (9), let us recall that singularities of the harmonic map flow have been divided by Struwe [8] into two types depending on whether the quantity $(-t)|\nabla u|^2$ remains bounded (type I) or not (type II) as $t \nearrow 0$ (here we assume, without loss of generality, that the blowup occurs at time $t=0$). Calculating this quantity for the equivariant ansatz (3) and (6), one finds that the blowup is of type I if and only if

$$f'(y)^2 + \frac{d-1}{y^2}\sin^2 f(y) < C \tag{10}$$

for some constant C and all $y \geq 0$. The condition (10) together with the requirement of smoothness is equivalent to the conditions (8) and (9). In the case of (8) this is evident. To see how (9) comes about, let us rewrite equation (7) in the integral form

$$f'(y) = \frac{d-1}{2} y^{1-d} e^{y^2/4} A(y), \quad A(y) = \int_0^y s^{d-3} e^{-s^2/4} \sin(2f(s))\, ds.$$

For $f'(y)$ to be bounded at infinity, it is necessary that $\lim_{y\to\infty} A(y) = 0$ and then (9) follows from l'Hôpital's rule. Thus, the conditions (8) and (9) are equivalent to the requirement that the blowup mediated by the shrinker (6) is of type I. □

One of the key ingredients of the shooting argument in [4] is that the linearized perturbations about the equator map $f = \pi/2$ are oscillating at infinity. This happens for $d^2 - 8d + 8 < 0$ which implies the upper bound $d = 6 = \lfloor 4 + 2\sqrt{2} \rfloor$ (of course, only integer values of d make sense geometrically). There is numerical evidence that there are no smooth shrinkers for $d \geq 7$; however, to our knowledge, this fact has not been proved. The aim of this note is to fill this gap by proving the following nonexistence result.

Theorem 1. For $d \geq 7$, there exists no smooth solution of equation (7) satisfying the conditions (8) and (9). □

Proof. The proof is extremely simple. Suppose that $f(y)$ is a global solution satisfying (8) and define the function $h(y) = y^3 f'(y)$. Multiplying equation (7) by y^2 and differentiating, we obtain

$$y^2 h'' = \alpha(y) h' + \beta(y) h, \tag{11}$$

where

$$\alpha(y) = \tfrac{1}{2}y(y^2 - 2d + 10) \quad \text{and} \quad \beta(y) = d - 7 + (d-1)(1 + \cos 2f). \tag{12}$$

We assume that $d \geq 7$, so $\beta(y) \geq 0$. It follows from (8) that $h(0) = h'(0) = h''(0) = 0$ and $h'''(0) = 6a > 0$, hence $h'(y) > 0$ for small y. We now show that $h'(y)$ cannot go to zero. Suppose otherwise and let y_0 be the first point at which $h'(y_0) = 0$. If $d > 7$ or $f(y_0) \neq \pi/2$, then $\beta(y_0) > 0$ and therefore $h''(y_0) = \beta(y_0)h(y_0) > 0$, contradicting that y_0 exists. If $d = 7$ and $f(y_0) = \pi/2$, then $\beta(y_0) = 0$ and $h''(y_0) = 0$, so a bit more work is needed. In this case, differentiating equation (11) we find that $h'''(y_0) = 0$ and differentiating once more we obtain $h^{(iv)}(y_0) = 24y_0^{-8}h^3(y_0) > 0$, again contradicting the existence of y_0. Thus, $h'(y) > 0$ for all y. From this, (11) and (12), we obtain

$$y^2 h''(y) = \alpha(y)h'(y) + \beta(y)h(y) > 0 \quad \text{for } y > \sqrt{2d - 10}. \tag{13}$$

Therefore, $\lim_{y\to\infty} h(y) = \infty$, contradicting (9). ■

We conclude with a few remarks.

Remark 2. Equation (7) is the Euler–Lagrange equation for the functional

$$\mathcal{E}(f) = \int_0^\infty \left(f'^2 + \frac{d-1}{y^2}\sin^2 f\right) y^{d-1}\, e^{-y^2/4}\, dy, \tag{14}$$

which can be interpreted as the Dirichlet energy for maps from $\mathbb{R}^d$ with the conformally flat metric $g = e^{-\frac{y^2}{2(d-2)}} g_{\text{flat}}$ into S^d. Thus, shrinkers can be viewed as harmonic maps from $(\mathbb{R}^d, g)$ into S^d. Note that $\mathcal{E}(f)$ is invariant under the reflection symmetry $f \to \pi - f$ and the equator map $f_e = \pi/2$ is the only fixed point of this symmetry. For this kind of functional Corlette and Wald conjectured in [3], using Morse theory arguments, that the number of critical points (counted without multiplicity) with energy below $\mathcal{E}(f_e)$ is equal to the Morse index of f_e (i.e., the number of negative eigenvalues of the Hessian of $\mathcal{E}$ at f_e). In the case of (14), the Morse index of f_e drops from infinity to two at $d = 4 + 2\sqrt{2}$ and then from two to one at $d = 7$ (see [1]). Thus, according to the conjecture of Corlette and Wald, for $d \geq 7$, there should be exactly one (modulo the reflection symmetry) critical point of $\mathcal{E}(f)$ (this unique critical point is, of course, $f = 0$), in perfect agreement with Theorem 1. □

Remark 3. Struwe showed that the type I singularities are asymptotically self-similar [7], that is their profile is given by a smooth shrinker. Therefore, Theorem 1 implies that in dimensions $d \geq 7$ all singularities for the equivariant harmonic map flow (4) must be of type II (see [1] for a recent analysis of such singularities). $\square$

Remark 4. It is well known that there are close parallels between the harmonic map and Yang–Mills heat flows [5]. For the spherically symmetric magnetic Yang–Mills potential $w(t,r)$ in $d \geq 3$ dimensions, a counterpart of equation (4) reads

$$w_t = w_{rr} + \frac{d-3}{r} w_r - \frac{d-2}{r^2} w(w-1)(w-2), \tag{15}$$

and a counterpart of equation (7) for shrinkers $w(t,r) = g(y)$ is

$$g'' + \left(\frac{d-3}{y} - \frac{y}{2}\right) g' - \frac{d-2}{y^2} g(g-1)(g-2) = 0. \tag{16}$$

The one-parameter families of local smooth solutions of this equation near the origin and near infinity satisfy

$$g(0) = g'(0) = 0, \quad g''(0) = a > 0, \tag{17}$$

and

$$g(\infty) = b, \quad \lim_{y\to\infty} y^3 g'(y) = -2(d-2)b(b-1)(b-2). \tag{18}$$

Using a similar shooting technique as in [4] one can easily show that for $5 \leq d \leq 9$ there are infinitely many shrinkers $g_n(y)$. One novel feature, in comparison with the harmonic map flow, is that the first shrinker is known explicitly [9]:

$$g_1(y) = \frac{y^2}{\gamma + \delta y^2}, \quad \gamma = \frac{1}{2}(6d - 12 - (d+2)\sqrt{2d-4}), \quad \delta = \frac{\sqrt{d-2}}{2\sqrt{2}}. \tag{19}$$

In complete analogy to Theorem 1, we have the following theorem.

Theorem 2. For $d \geq 10$, there exists no smooth solution of equation (16) satisfying the conditions (17) and (18). $\square$

Proof. The same as for Theorem 1. The only change is that now the function $h(y) = y^3 g'(y)$ satisfies equation (11) with different coefficients

$$\alpha(y) = \tfrac{1}{2} y(y^2 - 2d + 14) \quad \text{and} \quad \beta(y) = d - 10 + 3(d-2)(1-g)^2. \tag{20}$$

Note that for $d = 10$ the solution (19) becomes $g_1 = 1$ (which does not satisfy the regularity condition at the origin (17)), while for $d > 10$ the parameter γ is negative so $g_1(y)$ has a pole at $y = (-\gamma/\delta)^{1/2}$. $\blacksquare$

By arguments analogous to the ones given in Remarks 1 and 3, it follows that in dimensions $d \geq 10$ all singularities for the equivariant Yang–Mills flow (15) must be of type II.

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