Blowup for supercritical equivariant wave maps

Piotr Bizoń

Jagiellonian University and Albert Einstein Institute

joint work with Paweł Biernat (Bonn) and Maciej Maliborski (AEI)

IHES, 25 May 2016

Equivariant wave maps

- Wave map equation for $\phi : \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{S}^d \hookrightarrow \mathbb{R}^{d+1}$

$$\phi_{tt} - \Delta\phi + \left(|\phi_t|^2 - |\nabla\phi|^2\right) \phi = 0$$

- Our motivation: toy model of critical behavior for Einstein's equation.
- For equivariant maps of the form (where $r = |x|$)

$$\phi(t,x) = \left(\frac{x}{r} \sin u(t,r), \cos u(t,r)\right)$$

  the wave map equation reduces to

$$u_{tt} = u_{rr} + \frac{d-1}{r} u_r - \frac{d-1}{2r^2} \sin(2u)$$

- We want to understand global dynamics for smooth initial data $(u, u_t)|_{t=0}$.
- Basic question: do solutions remain smooth for all future times? If not, what is the mechanism of singularity formation (”blowup”)?

Preliminaries

- Conservation of energy

$$E(u) = \int_0^\infty \left(u_t^2 + u_r^2 + \frac{d-1}{r^2} \sin^2 u \right) r^{d-1} dr$$

- Smoothness at $r = 0$ implies that $u(t,0) = m\pi$ (we choose $m = 0$)
- Finiteness of energy implies that $u(t,\infty) = k\pi$ ($k \in \mathbb{Z}$). The degree k is preserved in evolution as long as the solution remains smooth.
- Scaling invariance: $u(t,r) \to u_\lambda(t,r) = u(t/\lambda, r/\lambda)$
- $E(u_\lambda) = \lambda^{d-2} E(u)$, hence $d = 2$ is critical and $d \geq 3$ are supercritical
- The critical dimension is well understood: B-Chmaj-Tabor '01, Struwe '03, Krieger-Schlag-Tataru '06, Sterbenz-Tataru '10, Ovchinnikov-Sigal '11, Raphaël-Rodnianski '12, Côte-Kenig-Lawrie-Schlag '12.
- Supercritical dimensions are underexplored. Few results for $d = 3$: Shatah '88, B-Chmaj-Tabor '00, Donninger '11, Donninger-Schörkhuber and, until recently, almost no results for $d \geq 4$.

Self-similar solutions

- Self-similar solutions are invariant under scaling $u(t/\lambda, r/\lambda) = u(t,r)$. Thus

$$u(t,r) = f(y) \quad \text{where} \quad y = \frac{r}{T-t}$$

- This gives an ODE

$$f'' + \left(\frac{d-1}{y} + \frac{(d-3)y}{1-y^2}\right) f' - \frac{d-1}{2y^2(1-y^2)} \sin(2f) = 0$$

- We want smooth solutions on $0 \le y \le 1$, the past light cone of $(T,0)$.
- For such solutions

$$u_r(t,0) = \frac{f'(0)}{T-t} \to \infty \quad \text{as} \quad t \nearrow T$$

- Remark: in order to participate in dynamics, self-similar solutions need to be smooth outside the light cone ($y > 1$) as well.

Self-similar solutions

- One-parameter family of local smooth solutions near the origin

$$f(y) = cy + \mathcal{O}(y^3)$$

- Local solutions extend smoothly to the whole interval $0 \leq y < 1$.
  For what values of c these solutions are smooth at $y = 1$?

- For $c_0 = \frac{2}{\sqrt{d-2}}$ the solution is known is closed form

$$f_0(y) = 2\arctan\left(\frac{y}{\sqrt{d-2}}\right)$$

  $d = 3$: Shatah '88, Turok-Spergel '90, $d \geq 4$: B-Biernat '15

- Conjecture: f_0 is the only self-similar solution for $d \geq 7$.

- Aside: harmonic map flow has no self-similar solutions for $d \geq 7$.

How f_0 was found?

- Let $\varepsilon = d-2$ and change variables $y=\sqrt{\varepsilon}\,x$ and $f(y)=\tilde{f}(x)$. Then

$$(1-y^2)f'' + \left(\frac{d-1}{y} - 2y\right)f' - \frac{d-1}{2y^2}\sin(2f) = 0$$

can be written in the form

$$\underbrace{\tilde{f}'' + \frac{1}{x}\tilde{f}' - \frac{\sin(2\tilde{f})}{2x^2}}_{=0 \text{ for } \tilde{f}_0 = 2\arctan(x)} = \varepsilon \underbrace{\left(x^2\tilde{f}'' - \left(\frac{1}{x} - 2x\right)\tilde{f}' + \frac{\sin(2\tilde{f})}{2x^2}\right)}_{\text{Miracle: } = 0 \text{ for } \tilde{f}_0\ !}$$

- Note that $f_0(1) < \pi/2$ for $d > 3$.

Self-similar solutions for $3 \le d \le 6$

If $f(y)$ is smooth at $y = 1$, then

$$(d-3)f'(1) - \frac{d-1}{2}\sin(2f(1)) = 0$$
$$(d-5)f''(1) + (d-7-(d-1)\cos(2f(1)))f'(1) = 0$$

This implies that

- For $d = 3$
$$f(y) = \frac{\pi}{2} - f'(1)(1-y) + ...$$
- For $d = 5$, either
$$f(y) = \frac{\pi}{2} + \frac{1}{2}f''(1)(1-y)^2 + ...$$
  or
$$f(y) = \frac{\pi}{3} - \frac{\sqrt{3}}{2}(1-y) + \frac{1}{2}f''(1)(1-y)^2 + ...$$
- For $d = 4, 6$
$$f(y) = f(1) - \frac{d-1}{2(d-3)}\sin(2f(1))(1-y) + ...$$

Self-similar solutions for $3 \le d \le 6$

Theorem

For each $d \in \{3,4,5,6\}$ there is an infinite sequence $(c_n)_{n\in\mathbb{N}}$ such that the corresponding solutions, denoted by $f_n(y)$, are smooth at $y = 1$.

Proof:

- Shooting argument for solutions with $f(1) = \pi/2$ in $d = 3,5$ [B '00]. Key ingredient: linearization around the singular solution $f = \pi/2$.
  In $d = 4,6$ the proof requires a minor modification (because $f(1) \neq \pi/2$).
- Self-similar solutions are (formally) critical points of the functional

$$\mathscr{E}(f) = \int_0^1 \left(f'^2 + \frac{d-1}{2} \frac{\sin^2 f - \sin^2 f(1)}{y^2(1-y^2)} \right) \frac{y^{d-1} dy}{(1-y^2)^{\frac{d-3}{2}}}$$

For $d = 5$ the variational proof of existence of $f_1(y)$ was given by Cazenave-Shatah-Tahvildar-Zadeh '98.

Spectral stability

- In terms of slow time $s = -\ln(T-t)$ and $U(s,y) = u(t,r)$ we have

$$U_{ss} + U_s + 2y\,U_{sy} = (1-y^2)U_{yy} + \left(\frac{d-1}{y} - 2y\right) U_y - \frac{d-1}{2y^2}\sin(2U)$$

- Inserting $U(s,y) = f_n(y) + e^{\lambda s} v(y)$ and linearizing we get the quadratic eigenvalue problem

$$(1-y^2)v'' + \left(\frac{d-1}{y} - 2(\lambda+1)y\right) v' - \lambda(\lambda+1)v - \frac{d-1}{y^2}\cos(2f_n)v = 0,$$

- We demand that $v \in C^\infty[0,1] \Rightarrow$ quantization of eigenvalues $\lambda_k^{(n)}$
- We conjecture that for each n the spectrum has the form

$$\underbrace{\cdots < \lambda_{-2}^{(n)} < \lambda_{-1}^{(n)}}_{\infty \text{ many stable modes}} < 0 < \underbrace{\lambda_0^{(n)} = 1}_{\text{gauge mode}} < \underbrace{\lambda_1^{(n)} < \cdots < \lambda_n^{(n)}}_{n \text{ unstable modes}}$$

- The eigenvalue $\lambda_0^{(n)} = 1$ corresponds to the gauge mode $v_0^{(n)}(y) = y f_n'(y)$ generated by the shift of the blowup time T.

Spectral stability of f_0

- In terms of new variables $x = \frac{(d-1)y^2}{y^2+d-2}$ and $v(y) = x^{1/2}\,(d-1-x)^{\frac{\lambda}{2}}\,w(x)$ the eigenvalue equation takes the form of the Heun equation

$$w'' + \left(\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\varepsilon}{x-d+1}\right) w' + \frac{\alpha\beta x - q}{x(x-1)(x-d+1)}\, w = 0$$

  where the coefficients $\gamma, \delta, \varepsilon, \alpha, \beta, q$ depend on d and λ.

- The analytic solution at $x=0$ is $w(x) = \sum\limits_{n=0}^{\infty} a_n x^n$, where

$$a_n \sim c_1(\lambda)\underbrace{n^{\lambda-\frac{d+1}{2}}}_{\text{bad}} + c_2(\lambda)\underbrace{(d-1)^{-n} n^{-\frac{3}{2}}}_{\text{good}} \qquad \text{for } n \to \infty$$

- The quantization condition $c_1(\lambda) = 0$ can be solved using continued fractions [B '05]. Recently, Costin-Donninger-Glogić '16 proved that $c_1(\lambda) = 0$ has no positive roots (apart from $\lambda = 1$).

Self-adjoint formulation

- Let $\psi(y) = (1-y^2)^{\lambda/2} y^{\frac{d-1}{2}} v(y)$. Then, the eigenvalue problem becomes

$$A_n \psi = \mu \psi, \qquad \mu = \lambda(d-1-\lambda)$$

  where the operator $A_n = -(1-y^2)^{\frac{d+1}{2}} \partial_y \left((1-y^2)^{\frac{d-3}{2}} \partial_y \right) + V(f_n(y))$ is self-adjoint on the Hilbert space $X = L^2 \left([0,1], (1-y^2)^{-\frac{d+1}{2}} dy \right)$.

- For $\lambda > \frac{d-1}{2}$, the eigenvalues of our problem (i.e. $v \in C^\infty[0,1]$) and the eigenvalues of A_n (i.e. $\psi \in X$) coincide.

- Using this correspondence and applying the Sturm oscillation theorem to the gauge mode $\psi_0^{(n)} = (1-y^2)^{1/2} y^{\frac{d+1}{2}} f_n'(y)$ with $\mu = d-2$, we conclude that f_n has n (for $d = 3, 4$) or $n-1$ (for $d = 5, 6$) eigenvalues $\lambda > d-2$.

- In addition, for $d = 5$ the gauge mode is the eigenfunction, hence $\lambda_1^{(n)} = 3$ is the eigenvalue for each $n \neq 0$.

- Numerical calculations indicate that for $d = 3, 4, 5$ there are no additional eigenvalues with positive real part, while for $d = 6$ there is exactly one such eigenvalue (which is *not* an eigenvalue of A_n).

Spectrum of eigenvalues for f_0 and f_1

$\lambda_k^{(0)}$	$k=0$	$k=-1$	$k=-2$	$k=-3$	$k=-4$
$d=3$	1	−0.542466	−2.000000	−3.398381	−4.765079
$d=4$	1	−0.563612	−2.109131	−3.603718	−5.061116
$d=5$	1	−0.572315	−2.163011	−3.711951	−5.216059
$d=6$	1	−0.577089	−2.195673	−3.780281	−5.306294
$d=7$	1	−0.580109	−2.217711	−3.827722	−5.354120
$d=8$	1	−0.582193	−2.233621	−3.862716	−5.367078

$\lambda_k^{(1)}$	$k=1$	$k=0$	$k=-1$	$k=-2$	$k=-3$
$d=3$	6.333625	1	−0.518609	−1.743834	−2.867543
$d=4$	3.998831	1	−0.390210	−1.585419	−2.714684
$d=5$	3	1	−0.281770	−1.447552	−2.574483
$d=6$	2.426239	1	−0.179962	−1.308475	−2.419907

Self-similar solutions as attractors

Conjecture (for all $d \geq 3$)

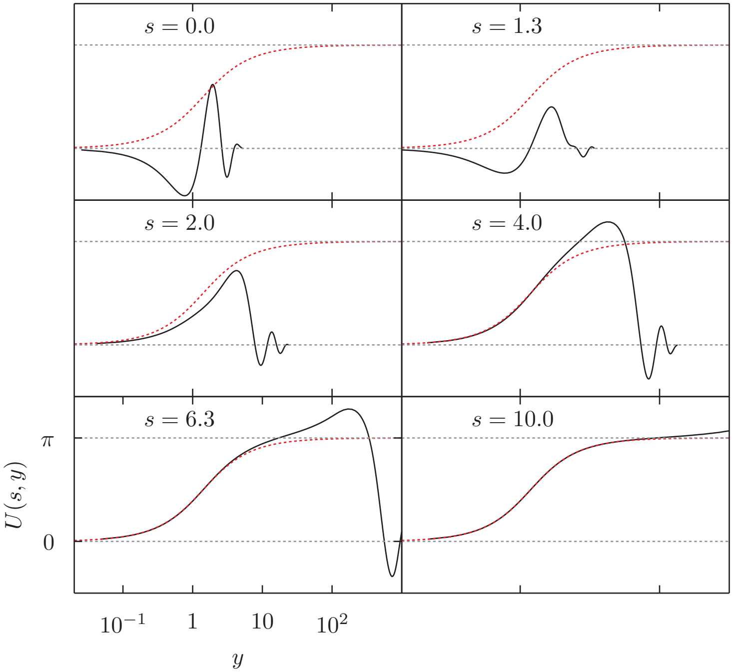
The self-similar solution f_0 is a universal attractor for generic blowup, i.e. if a solution $u(t,r)$ blows up at time T, then $\lim_{t \nearrow T} u(t,(T-t)r) = f_0(r)$.

Evidence:

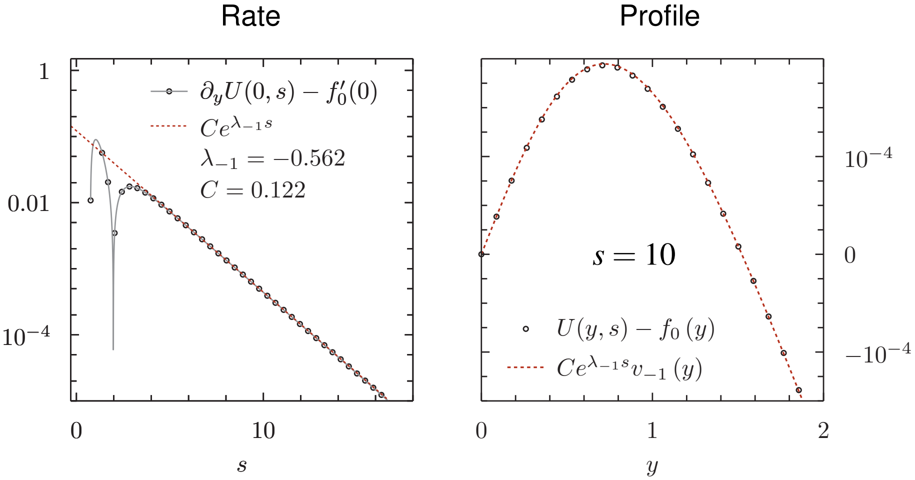
- For $d = 3$ Donninger '11 proved that the spectral stability of f_0 implies its linear and nonlinear stability. An extension of this result to higher dimensions seems feasible but the non-perturbative regime seems hard.
- Numerical studies: first done for $d = 3$ [B-Chmaj-Tabor '00], recently have been extended to higher dimensions [Biernat-B-Maliborski '16]. They confirm the above conjecture and verify that the rate and profile of convergence to f_0 are determined by the least damped mode

$$u(t,r) - f_0\left(\frac{r}{T-t}\right) \sim C\,(T-t)^{-\lambda_{-1}}\, v_{-1}\left(\frac{r}{T-t}\right),$$

where the coefficient C and blowup time T depend on initial data.



s = 0.0
s = 1.3
s = 2.0
s = 4.0
s = 6.3
s = 10.0
π
0
U(s, y)
10^−1
1
10
10^2
y



Excellent quantitative agreement with the linear approximation

$$U(s,y) = f_0(y) + C\, e^{\lambda_{-1}}\, v_{-1}(y) + \dots$$

Threshold of blowup

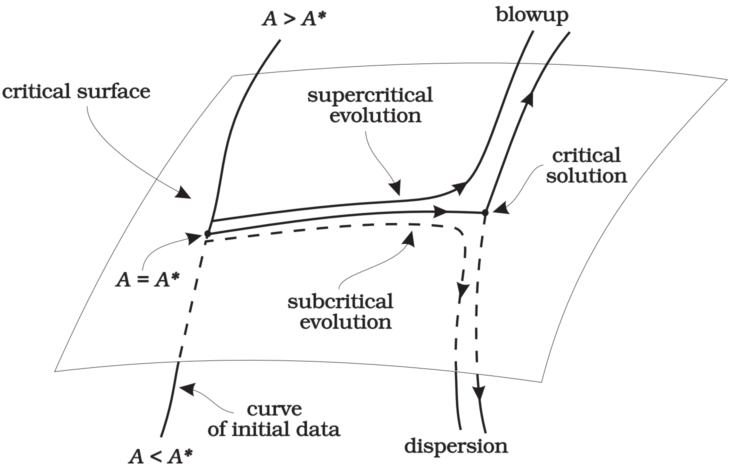
- Small solutions disperse and large solutions blow up. What is the borderline between these two generic outcomes of evolution?
- Basic numerical technique: consider a curve of initial data that interpolates between small and large data, say a gaussian with amplitude A. Using bisection, one can fine tune to critical amplitude A^*.
- In dimensions $3 \leq d \leq 6$ the evolution of marginally critical data exhibits a typical saddle-point behavior for intermediate times

$$U(s,y) \simeq f_1(y) + c_1(A - A^*)e^{\lambda_1 s} v_1(y) + c_{-1} e^{\lambda_{-1} s} v_{-1}(y) + \dots$$

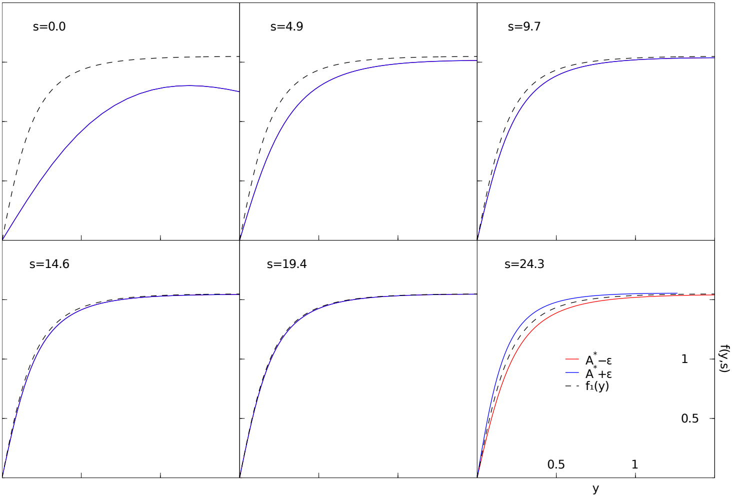
  where $\lambda_1 > 0$ and $\lambda_{-1} < 0$.
- For dispersive solutions this implies that $\max |u_r(t,0)| \sim |A^* - A|^{-1/\lambda_1}$

Conjecture (for $3 \leq d \leq 6$)

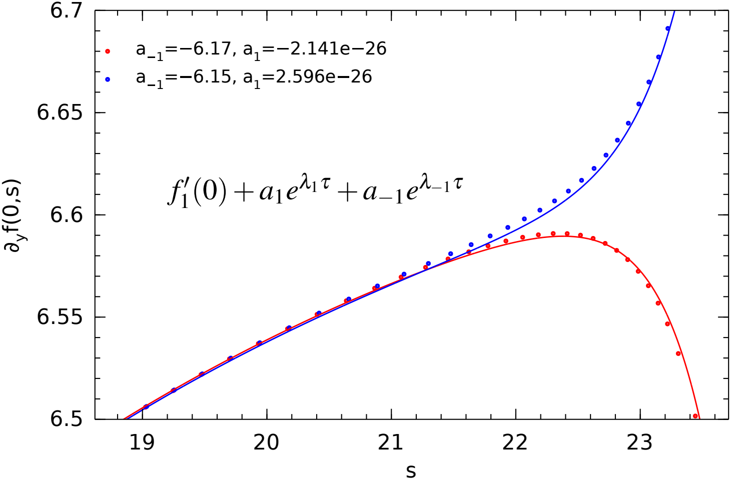
The self-similar solution f_1 plays the role of the critical solution whose codimension-one stable manifold separates blowup from dispersion.



Schematic picture of evolution near the threshold.



Two marginally critical solutions with $A = A^* \pm 10^{-26}$



6.7
6.65
6.6
6.55
6.5
19
20
21
22
23
s
$\partial_y f(0,s)$
a_{-1}=−6.17, a_1=−2.141e−26
a_{-1}=−6.15, a_1=2.596e−26
$f_1'(0) + a_1 e^{\lambda_1 \tau} + a_{-1} e^{\lambda_{-1} \tau}$

Threshold of blowup in $d \geq 7$ (à la Herrero-Velazquez)

- For $d \geq 7$ the singular solution $f = \pi/2$ has spectrum ($k = 0, 1, ...$)

$$\lambda_k = \gamma - k, \qquad \gamma = \frac{1}{2}\left(d - 2 - \sqrt{d^2 - 8d + 8}\right)$$

  $\lambda_0 > 0$ is the gauge mode, $\lambda_1 > 0$, and $\lambda_k \leq 0$ for $k \geq 2$.
- Outer solution: $f_{out} = \pi/2 + \cancel{a_1 e^{\lambda_1 s} v_1(y)} + a_2 e^{\lambda_2 s} v_2(y) + \ldots$
- Inner solution: $f_{in} = F(r/\alpha(t))$, where $F(r)$ is the smooth static solution, i.e. $F'' + \frac{d-1}{r} F' - \frac{d-1}{2r^2} \sin(2F) = 0$ with $F(r) \sim r$ for $r \to 0$.
- Since $v_2(y) \sim y^{-\gamma}$ for $y \to 0$ and $F(r) - \pi/2 \sim r^{-\gamma}$ for $r \to \infty$, we can match f_{out} and f_{in} in the intermediate region. This yields

$$\alpha(t) \sim (T - t)^{\beta}, \qquad \beta = 1 - \lambda_2/\gamma = 2/\gamma > 1$$

- For $d = 7$ the above analysis breaks down because $\lambda_2 = 0$.
- New approach to Type II blowup due to Merle-Raphaël-Rodnianski '14 in the context of supercritical NLS (adapted to the supercritical wave equation by Collot '14) seems applicable here (Biernat, in progress).

Selected open problems

- Threshold of blowup in $d = 2$: blowup has a universal form of shrinking harmonic map $u(t,r) \sim 2\arctan\left(\frac{r}{\alpha(t)}\right)$ with $\alpha(t) \to 0$ for $t \nearrow T$ [Struwe'03]. For stable blowup $\alpha(t) \sim C(T-t)e^{-\sqrt{|\ln(T-t)|}}$ [Ovchinnikov-Sigal '11, Raphaël-Rodnianski '12]. What is the speed of blowup at the threshold?
- Continuation beyond blowup: we expect that a solution that blows up along f_0 at time T_1 immediately recovers smoothness for $T > T_1$ and remains smooth until (possibly) the next blowup occurs.
- Blowup for wave maps on confined geometries: blowup does not depend on the geometry of domain but the very occurrence of blowup does. Our preliminary results for wave maps from AdS_4 to S^3 suggest that for 'generic' small smooth initial data of size ε the time of blowup $T \sim \varepsilon^{-2}$.
- Supercritical Einstein-wave-map system: Extremely rich phenomenology depending on the dimensionless parameter $\kappa = G\beta^2$. Generic self-similar blowup (for small κ) disappears for large κ (gravitational regularization) and there appears a codimension-one discretely-self similar solution.